Hardy inequalities on the Heisenberg group for convex bounded polytopes

Bartosch Ruszkowski

joint work with Hynek Kovařík and Timo Weidl

University of Stuttgart

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Hardy inequalities for the Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain for $n \in \mathbb{N}$. Let $4 \leq c(\Omega)$ be the smallest constant such that

$$\int_\Omega \frac{|u(x)|^2}{\mathrm{dist}(x, \partial\Omega)^2} \, \mathrm{d}x \;\leq\; c(\Omega) \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x,$$

holds for all $u \in C_0^\infty(\Omega)$, where

$$\mathrm{dist}(\mathrm{x}, \partial\Omega) := \inf_{\mathrm{x} \in \partial\Omega} \|x - y\|_{\mathrm{e}}.$$

- If Ω is convex, then $c(\Omega) = 4$.
- If $\Omega \subset \mathbb{R}^2$ is simply connected, then $c(\Omega) \leq 16$.
- If Ω is a Lipschitz domain, $c(\Omega) < \infty$.

The Heisenberg group

The first Heisenberg is is defined as $\mathbb{R}^3$ equipped with the group law

$$
\begin{aligned}
&(x_1, x_2, x_3) \boxplus (y_1, y_2, y_3) \\
:= \;& \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\Big).
\end{aligned}
$$

The left-invariant vector fields at the point $x := (x_1, x_2, x_3) \in \mathbb{H}$;

$$X_1 \; := \; \partial_{x_1} + \frac{x_2}{2}\partial_{x_3}, \quad X_2 \; := \; \partial_{x_2} - \frac{x_1}{2}\partial_{x_3}, \quad X_3 := \partial_{x_3}$$

The only non-vanishing commutation relation is

$$[X_1, X_2] = -X_3.$$

The sub-gradient is denoted by

$$\nabla_{\mathbb{H}} := (X_1, X_2).$$

We call a Lipschitz curve $\gamma : [a,b] \to \mathbb{H}$ parametrized by $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ **horizontal** if for all $t \in [a,b]$ holds

$$\gamma'(t) = \alpha(t) X_1(\gamma(t)) + \beta(t) X_2(\gamma(t)),$$

The **Carnot-Carathéodory** distance between x and y is then defined as

$$\begin{aligned} d_C(x,y) &:= \inf_{\gamma} \int_a^b \sqrt{\alpha(t)^2 + \beta(t)^2} \;\, \mathrm{d}t \\ &= \inf_{\gamma} \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} \;\, \mathrm{d}t, \end{aligned}$$

where the infimum is taken over all horizontal curves γ connecting x and y.

The Hardy inequality on $\mathbb{H}$

Let $\Omega \subset \mathbb{H}$ be a bounded domain and $4 \leq c(\Omega)$ such that

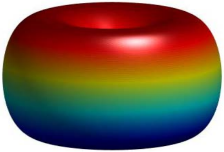
$$\int_{\Omega} \frac{|u(x)|^2}{\delta_C(x)^2} \, \mathrm{d}x \;\leq\; c(\Omega) \int_{\Omega} |\nabla_{\mathbb{H}} u(x)|^2 \, \mathrm{d}x$$

holds for all $u \in C_0^\infty(\Omega)$, where

$$\delta_C(x) := \inf_{y \in \partial\Omega} d_C(x, y).$$

$d_C(x, y)$ is the Carnot-Carathéodory distance between x and y.

- If $\Omega = \{x \in \mathbb{H} \mid d_C(x, 0) < r\}$ Yang (2013) showed, that $c(\Omega) = 4$ for any $r > 0$.



- If Ω has $C^{1,1}$ regular boundary Danielli, Garofalo and Phuc (2009) showed, that $c(\Omega) < \infty$.

Results

Theorem

Let $\Omega \subset \mathbb{H}$ be an open bounded convex polytope and let $m \in \mathbb{N}$ the number of hyperplanes of $\partial\Omega$, which are not orthogonal to the hyperplane $x_3 = 0$. Then holds for all $u \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \;\leq 5\,(c_m+1)^{4/3}\int_\Omega |\nabla_{\mathbb{H}}\,u(x)|^2\,\mathrm{d}x,$$

where

$$c_m = m^{8/9}\pi\,3^{5/2}\,2^{8/3}\left(1+\frac{1}{12\sqrt{6}}\right)^{2/3},$$

Lemma

Let Ω be an open bounded domain in $\mathbb{H}$. Then holds

$$\begin{aligned} \int_\Omega \frac{|u(x)|^2}{d_1(x)^2}\,\mathrm{d}x &\leq 4\int_\Omega |X_1\,u(x)|^2\,\mathrm{d}x, \\ \int_\Omega \frac{|u(x)|^2}{d_2(x)^2}\,\mathrm{d}x &\leq 4\int_\Omega |X_2\,u(x)|^2\,\mathrm{d}x \end{aligned}$$

for all $u \in C_0^\infty(\Omega)$, where the distances $d_1(x)$ and $d_2(x)$ are given by

$$\begin{aligned} d_1(x) &:= \inf_{s\in\mathbb{R}} \{|s| > 0 \,|\, x + s(1, 0, x_2/2) \notin \Omega\}, \\ d_2(x) &:= \inf_{s\in\mathbb{R}} \{|s| > 0 \,|\, x + s(0, 1, -x_1/2) \notin \Omega\}. \end{aligned}$$

We define the Kaplan gauge for $x := (x_1, x_2, x_3) \in \mathbb{H}$

$$\|x\|_{\mathbb{H}}^4 := (x_1^2 + x_2^2)^2 + 16x_3^2 .$$

For all $x, y \in \mathbb{H}$ it holds

$$\frac{1}{\pi} d_C(x,y)^2 \leq \|(-y) \boxplus x\|_{\mathbb{H}}^2 \leq d_C(x,y)^2.$$

Moreover, both inequalities are sharp.

Theorem (Goldstein, Kombe (2007))

For all $u \in C_0^\infty(\mathbb{H})$ holds

$$\int_{\mathbb{H}} \frac{|u(x)|^2}{d_C(x,0)^2} \,\mathrm{d}x \;\leq\; \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x.$$

Immediately holds for all $a \in \mathbb{H}$

$$\frac{1}{\pi} \int_{\mathbb{H}} \frac{|u(x)|^2}{\|(-a) \boxplus x\|_{\mathbb{H}}^2} \,\mathrm{d}x \;\leq\; \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x.$$

The Proof

Let Ω be a bounded convex polytope and let $m \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_m$ be the charateristic points of $\partial\Omega$.

$$5 \int_\Omega |\nabla_{\mathbb{H}} u(x)|^2 \, \mathrm{d}x$$

$$\geq \int_\Omega \left(\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} + \frac{1}{\pi m} \sum_{j=1}^{m} \frac{1}{\|(-a_j) \boxplus x\|_{\mathbb{H}}^2} \right) |u(x)|^2 \, \mathrm{d}x.$$

Let us consider $b \in \partial\Omega$ such that

$$\|(-b) \boxplus x\|_{\mathbb{H}} = \inf_{y \in \partial\Omega} \|-y \boxplus x\|_{\mathbb{H}}.$$

There are two classes of hyperplanes, which are equivalent by left-translation to each other

- The hyperplane orthogonal to $y_3 = 0 \Leftrightarrow n_3 = 0$.
  No characteristic points are contained.
- The hyperplanes not orthogonal to $y_3 = 0 \Leftrightarrow n_3 \neq 0$.
  A **unique** characteristic point is contained.

The Proof

For $b \in \{y \in \mathbb{H} \mid y_3 = 0\}$ we have the lower bound

$$\frac{x_1^2 + x_2^2}{4x_3^2} + \frac{1}{\pi m} \frac{1}{\sqrt{(x_1^2 + x_2^2)^2 + 16x_3^2}}.$$

Proposition

Let $x \in \mathbb{H}$ and $a > 0$. For the case $x_1^2 + x_2^2 \leq a|x_3|$ it holds

$$\left(\inf_{y_3 = 0} \| -y \boxplus x \|_{\mathbb{H}} \right)^{-2} \leq \frac{4 \cdot 3^3}{|x_3|} \left(1 + \frac{a}{48\sqrt{6}} \right)^2$$

and for $x_1^2 + x_2^2 \geq a|x_3|$ it holds

$$\left(\inf_{y_3 = 0} \| -y \boxplus x \|_{\mathbb{H}} \right)^{-2} \leq \frac{x_1^2 + x_2^2}{4x_3^2} \left(\frac{48\sqrt{6}}{a} + 1 \right)^{-4/3}.$$

Theorem

Let $\Omega = \omega \times (0,1)$ such that and $\omega \subset \mathbb{R}^2$ is a bounded convex domain. For fixed $a > 0$ we assume that for all $x \in \Omega$ holds

$$x_1^2 + x_2^2 \geq a|x_3|, \quad \textit{and} \quad x_1^2 + x_2^2 \geq a|-1 + x_3|.$$

Then holds for all $u \in C_0^\infty(\Omega)$

$$\left(\frac{48\sqrt{6}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \;\leq 4 \int_\Omega |\nabla_{\mathbb{H}} u(x)|^2\,\mathrm{d}x.$$

Example: The set $\Omega_a := B_1(p_a) \times (0,1)$, where $B_1(p_a)$ is the two-dimensional Euclidean ball with radius one centered at $p_a := (\sqrt{a} + 1, 0)$.

Thank you for the attention!