NLS on graphs: recent results and perspectives

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Physical Background

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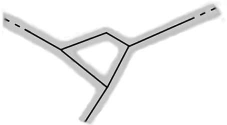

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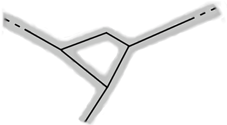

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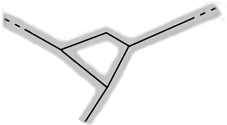

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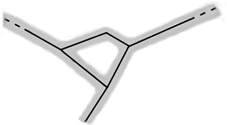

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Note: experimental evidence for BEC in ramified traps (e.g. Tokuno et al. '08, Hung et al. '11, Lorenzo et al. '14).

Basics on metric graphs

A metric graph is a multigraph $\mathcal{G} = (\mathrm{V}, \mathrm{E})$, where each edge e joining two vertices v_1 and v_2 is associated either with a closed bounded interval

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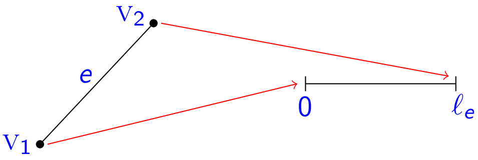

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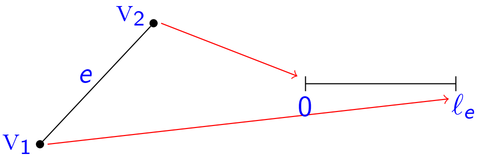

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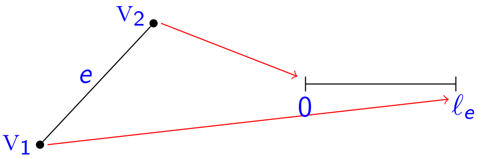

Note: half–lines are always attached to the graph at $x_e = 0$.

Examples

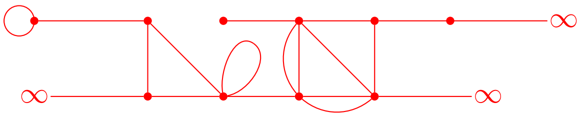

Fig.1: 19 edges (2 self-loops, 2 multiple, 3 unbounded), 13 vertices (1 of degree two, 3 at infinity).

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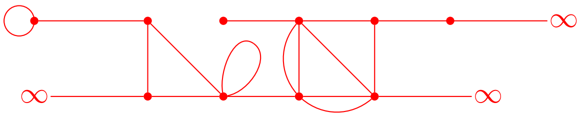

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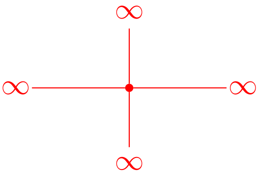

Fig.2: 4 edges (unbounded), 5 vertices (4 at infinity).

Functions on graphs

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and $H^1(\mathcal{G})$ as the set of continuous functions u (continuity means no jumps at vertices) such that

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Note: we focus on the generalized issue, with 4 replaced by $p \geq 2$.

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When $\mathcal{G} = \mathbb{R}$, for $p \in (2, 6)$ and $\mu > 0$, ground states exist and are translates of the soliton, namely $\varphi_\mu : \mathbb{R} \to \mathbb{R}^+$ such that

$$\varphi_\mu(x) = C\mu^{\frac{2}{6-p}} \operatorname{sech}^{\frac{2}{p-2}}(c\mu^{\frac{p-2}{6-p}} x),$$

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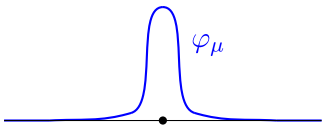

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When $\mathcal{G} = \mathbb{R}^+$, for $p \in (2,6)$ and $\mu > 0$, there is exactly one (positive) ground state given by half a soliton, namely $\widehat{\varphi}_\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that

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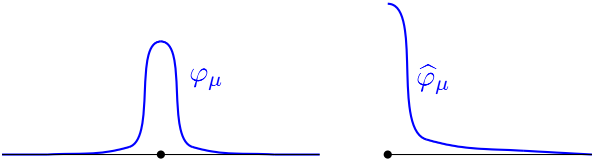

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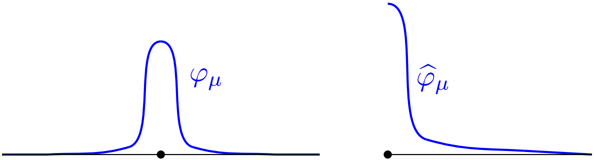

Note: one can prove that $\mathcal{E}(\widehat{\varphi}_\mu) < \mathcal{E}(\varphi_\mu) < 0$.

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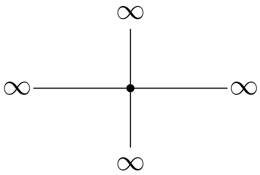

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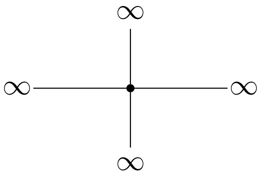

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then for $p \in (2,6)$ and $\mu > 0$

$$\inf_{u \in M} \mathcal{E}(u) = \mathcal{E}(\varphi_\mu),$$

but the infimum is not attained $\Rightarrow$ no ground state (Adami, Cacciapuoti, Finco, Noja '12).

Abstract results

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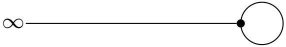
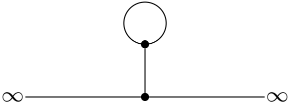


Tadpole graph



Signpost graph

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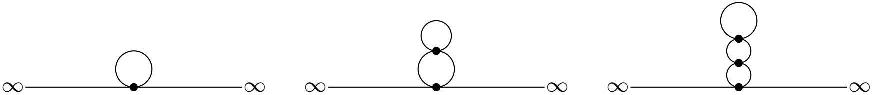

Some towers of bubbles

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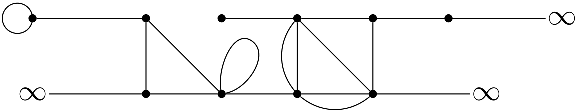

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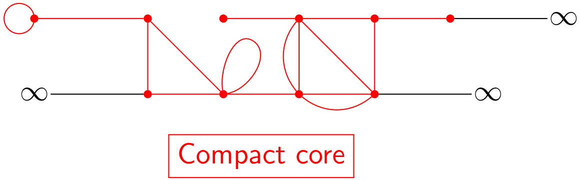

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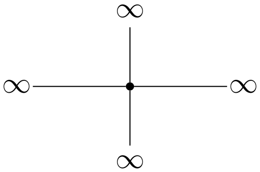

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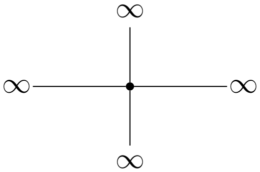

Compact core $= \emptyset$

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$$M = \{v \in H^1(\mathcal{G}) : \|v\|^2_{L^2(\mathcal{G})} = \mu\}.$$

Then, the issue of the ground states of mass μ reads:

(P)

to find functions $u \in M$ such that

$$E_M(u) = \inf_{v \in M} E_M(v).$$

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Bound states for localized nonlinearities

However, one could be also interested in stationary solutions $\psi(t,x) = e^{i\lambda t} u(x)$ of Gross–Pitaevskii with nonlinearity localized on $\mathcal{K}$, which are not necessarily minimizers of E_M, that is bound states.

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(ii) for every vertex v in $\mathcal{K}$

$$\sum_{e \succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0 \quad \text{(Kirchhoff condition)},$$

where "$e \succ \mathrm{v}$" means that e is incident at v and $du_e/dx_e(\mathrm{V})$ stands for $u_e'(0)$ or $-u_e'(\ell_e)$ depending on the orientation of I_e.

Existence and multiplicity

Theorem (Serra, T. – JDE, 2016)

For every $k \in \mathbb{N}$, there exists $\widetilde{\mu}_k > 0$ such that for all $\mu \geq \widetilde{\mu}_k$ there exist at least k distinct pairs $(\pm u_j)$ of bound states of mass μ. Moreover, for every $j = 1, \dots, k$

$$E_M(\pm u_j) \leq j\mathcal{E}(\varphi_{\mu/j}) + \sigma_k(\mu) < 0$$

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Note: other results on bound states (for a tadpole graph) can be found in Cacciapuoti, Finco, Noja '15, Noja, Pelinovsky, Shaikhova '15.

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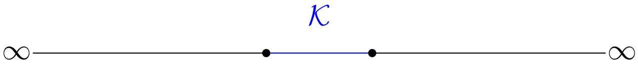

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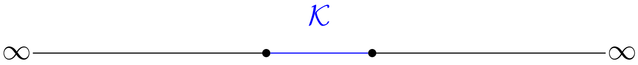

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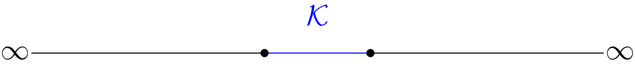

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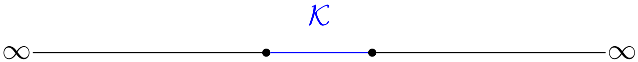

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Nonlinearity on a "compact portion of positive measure" generates bound states at higher energies!

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5. NLS on multidimensional structures, as for instance simplicial complexes.

THANK YOU FOR YOUR

ATTENTION!

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$$\|u\|_{L^p(\mathcal{G})}^p \leq \mathcal{C}_p \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1} \quad \forall u \in H^1(\mathcal{G}),$$

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there results that each minimizing sequence $(u_k) \subset M$ is bounded in $H^1(\mathcal{G})$.

Then $u_k \rightharpoonup u$ in $H^1(\mathcal{G})$ and $u_k \to u$ in $L^p(\mathcal{G})$, so that

$$E(u) \leq \liminf_k E_M(u_k).$$

Since $\inf_{v \in M} E_M(v) < 0$ prevents $\|u\|_{L^2(\mathcal{G})}^2 < \mu$, u solves (P).

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For $\alpha \in (0, \sqrt{\mu/L})$ and $m = \frac{\mu - \alpha^2 L}{N}$, consider the competitor u defined by

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Note: the competitor u is never a minimizer.

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Since $\inf_{v \in M} E_M(v) \leq 0$ for all $\mu > 0$, it is sufficient to find $\mu_2 > 0$ such that for all $\mu < \mu_2$ and all $u \in M = \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\}$ there results

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By an inductive argument one sees that $E_M(u) \leq 0$ entails

$$\|u'\|_{L^2(\mathcal{G})^2} \leq \frac{1}{\mathcal{C}^4_\infty \mu} \|u\|^{4(\frac{p}{4})^{n+1}}_{L^\infty(\mathcal{G})} (\mathcal{C}^4_\infty \mu L)^{\sum_{i=1}^n (\frac{p}{4})^i} \quad \forall n \geq 0,$$

by a repeated use of the L^∞ version of the Gagliardo–Nirenberg inequality

$$\|u\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}_\infty \|u\|^{1/2}_{L^2(\mathcal{G})} \|u'\|^{1/2}_{L^2(\mathcal{G})} \quad \forall u \in H^1(\mathcal{G}).$$

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then $\|u'\|^2_{L^2(\mathcal{G})} = 0$. Since $u \in H^1(\mathcal{G})$, there follows that $u \equiv 0$, but this is a contradiction with $\|u\|^2_{L^2(\mathcal{G})} = \mu > 0 \Rightarrow$ ~~$E_M(u) \leq 0$~~.

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Then, define the min–max levels

$$c_j = \inf_{A \in \Gamma_j} \max_{u \in A} E_M(u),$$

where $\Gamma_j = \{A \subset M : A \text{ symmetric, compact, } \gamma(A) \geq j\}$.

Sketch of proof: bound states

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if $c_j \in \mathbb{R}$ and E_M satisfies the Palais–Smale condition at level c_j, then c_j is a critical level for E_M.

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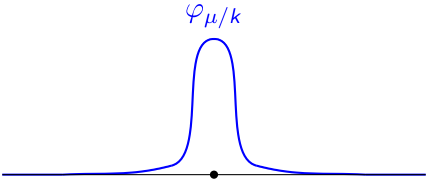
Since this entails $c_k < 0$ and, by definition, $c_1 \leq c_2 \leq \cdots \leq c_k$.

Sketch of proof: bound states



For fixed k, consider a soliton of mass μ/k

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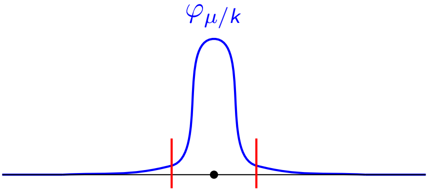

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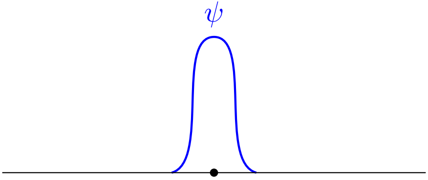

For fixed k, consider a soliton of mass μ/k, cut–off its "tails", lower it and arrange the mass ($\|\psi\|^2_{L^2(\mathbb{R})} = \mu/k$).

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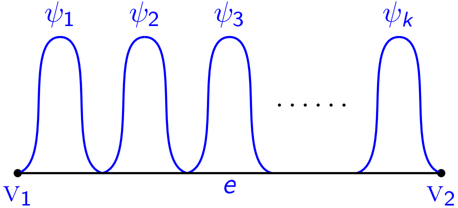

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Now place k copies of ψ on an edge of $\mathcal{K}$ and define $h : S^{k-1} \to M$ as

$$h(\theta) = \sqrt{k}\,(\theta_1\psi_1 + \theta_2\psi_2 + \cdots + \theta_k\psi_k).$$

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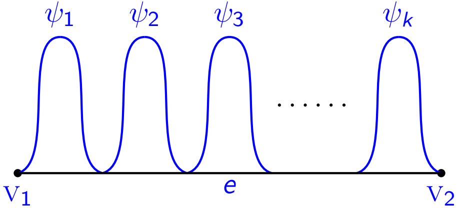

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Then, one can check that the required set is given by

$$\boxed{A = h(S^{k-1}).}$$