Geometric description of finite-dimensional quantum systems in complex projective spaces and applications

Davide Pastorello

University of Trento and INFN

Bressanone, 9 February 2016

Outline

1. *Hamiltonian geometric formulation of QM*
   → Complex projective space as quantum phase space;
   → Observables and density matrices as phase space functions;
   → Quantum dynamics as a Hamiltonian flow.

2. *Composite systems and entanglement*
   → A geomteric entanglement measure.

3. *Quantum control theory*
   → Study of quantum controllability with classical machinery.

Classical tools

Phase space

A classical system with n spatial degrees of freedom is described in a $2n$-**dimensional symplectic manifold** $(\mathcal{M}, \omega)$.

Dynamics

Hamilton equation

$$\frac{dx}{dt} = X_H(x(t))$$

Liouville equation

$$\frac{\partial \rho}{\partial t} + \{\rho, H\}_{PB} = 0$$

$H : \mathcal{M} \to \mathbb{R}$ is the *Hamiltonian function*.
X_H is the *Hamiltonian vector field*, given by: $\omega(X_H, \cdot) = dH(\cdot)$

Classical expecation values of $f : \mathcal{M} \to \mathbb{R}$

$$\langle f \rangle_\rho = \int_{\mathcal{M}} f(x)\rho(x)d\mu(x)$$

Observable C*-algebra

$$\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$$

QM in a classical-like fashion

Standard formulation of QM in a Hilbert space $\mathcal{H}$:

Quantum states: $D = \{\sigma \in \mathfrak{B}_1(\mathcal{H}) | \sigma \geq 0, tr(\sigma) = 1\}$
Quantum observables: Self-adjoint operators in $\mathcal{H}$.

Pure states (extremal points of D) are in bijective correspondence with projective rays in $\mathcal{H}$:

$$\mathcal{P}(\mathcal{H}) = \frac{\mathcal{H}}{\sim} \setminus [0] \qquad \psi \sim \phi \ \Leftrightarrow \exists \alpha \in \mathbb{C} \setminus \{0\} \ s.t. \ \psi = \alpha\phi$$

QM in a classical-like fashion

Standard formulation of QM in a Hilbert space $\mathcal{H}$:

Quantum states: $D = \{\sigma \in \mathfrak{B}_1(\mathcal{H}) | \sigma \geq 0, tr(\sigma) = 1\}$
Quantum observables: Self-adjoint operators in $\mathcal{H}$.

Pure states (extremal points of D) are in bijective correspondence with projective rays in $\mathcal{H}$:

$$\mathcal{P}(\mathcal{H}) = \frac{\mathcal{H}}{\sim} \setminus [0] \qquad \psi \sim \phi \;\Leftrightarrow \exists \alpha \in \mathbb{C} \setminus \{0\} \; s.t. \; \psi = \alpha\phi$$

$\dim\mathcal{H} = n < +\infty$
$\mathcal{P}(\mathcal{H})$ is a real $(2n-2)$-dimensional manifold with the following characterization of tangent space:

$p \in \mathcal{P}(\mathcal{H})$: $\forall v \in T_p\mathcal{P}(\mathcal{H}) \; \exists A_v \in i\mathfrak{u}(n)$ s.t. $v = -i[A_v, p]$.

$\mathfrak{u}(n)$ is the Lie algebra of $U(n)$

QM in a classical-like fashion

Geometry of $\mathcal{P}(\mathcal{H})$

Symplectic form: $\omega_p(u,v) := -i\,k\,tr([A_u, A_v]p) \qquad k > 0.$

Riemannian metric:
$g_p(u,v) := -k\,tr(([A_u,p][A_v,p] + [A_v,p][A_u,p])p) \qquad k > 0$

Complex form: $j_p : T_p\mathcal{P}(\mathcal{H}) \ni v \mapsto i[v,p] \in T_p\mathcal{P}(\mathcal{H})$
$p \mapsto j_p$ is smooth and $j_p j_p = -id$ for any $p \in \mathcal{P}(\mathcal{H})$:

$$\omega_p(u,v) = g_p(u, j_p v)$$

$(\mathcal{P}(\mathcal{H}), \omega, g, j)$ is a Kähler manifold.

QM in a classical-like fashion

Quantum observables as phase space functions

$\mathcal{O} : i\mathfrak{u}(n) \ni A \mapsto f_A : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$

Quantum states as Liouville densities

$\mathcal{S} : D \ni \sigma \mapsto \rho_\sigma : \mathcal{P}(\mathcal{H}) \to [0, 1]$

Set up a Hamiltonian theory

-) Equivalence Hamilton/Schrödinger dynamics:

$$\frac{dp}{dt} = -i[H, p(t)] \quad \Leftrightarrow \quad \frac{dp}{dt} = X_{f_H}(p(t))$$

-) Equivalence of expectation values:

$$\langle A \rangle_\sigma = tr(A\sigma) = \int_{\mathcal{P}(\mathcal{H})} f_A \, \rho_\sigma d\mu$$

From operators to functions

Definition

A map $f : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ is called **frame function** if there is $W_f \in \mathbb{C}$ s.t.

$$\sum_{p \in N} f(p) = W_f$$

$\forall N \subset \mathcal{P}(\mathcal{H})$ such that $d_g(p_1, p_2) = \frac{\pi}{2}$ for $p_1, p_2 \in \mathcal{P}(\mathcal{H})$ with $p_1 \neq p_2$ and N is maximal w.r.t. this property.

From operators to functions

Definition

A map $f : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ is called **frame function** if there is $W_f \in \mathbb{C}$ s.t.

$$\sum_{p \in N} f(p) = W_f$$

$\forall N \subset \mathcal{P}(\mathcal{H})$ such that $d_g(p_1, p_2) = \frac{\pi}{2}$ for $p_1, p_2 \in \mathcal{P}(\mathcal{H})$ with $p_1 \neq p_2$ and N is maximal w.r.t. this property.

Observable C*-algebra in terms of phase space functions

(V.Moretti, D.P. 2014)

$$\mathcal{F}^2(\mathcal{H}) := \{f : \mathcal{P}(\mathcal{H}) \to \mathbb{C} | \, f \in \mathcal{L}^2(\mathcal{P}(\mathcal{H}), \mu), \; f \text{ is a frame function}\}$$

$$\mathcal{O} : i\mathfrak{u}(n) \ni A \mapsto f_A \qquad f_A(p) = k tr(Ap) + \frac{1-k}{n} tr(A) \quad k > 0$$

$$\mathcal{S} : D \ni \sigma \mapsto \rho_\sigma \qquad \rho_\sigma(p) = \frac{n(n+1)}{k} tr(\sigma p) + \frac{k-(n+1)}{k}$$

Composite quantum systems

Composite system described in $\mathcal{H}_1 \otimes \mathcal{H}_2$

The phase space is $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and not $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$... But:

$\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$ is embedded in $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by Segre embedding:

$$Seg(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = |\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2|$$

and $Seg^* \left(\mathcal{F}^2(\mathcal{H}_1 \otimes \mathcal{H}_2)\right) = \mathcal{F}^2(\mathcal{H}_1) \otimes \mathcal{F}^2(\mathcal{H}_2)$.

Composite quantum systems

Composite system described in $\mathcal{H}_1 \otimes \mathcal{H}_2$

The phase space is $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and not $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$... But:

$\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$ is embedded in $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by Segre embedding:

$$Seg(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = |\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2|$$

and $Seg^* \left(\mathcal{F}^2(\mathcal{H}_1 \otimes \mathcal{H}_2)\right) = \mathcal{F}^2(\mathcal{H}_1) \otimes \mathcal{F}^2(\mathcal{H}_2)$.

Measure of entanglement

Let $\rho : \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to [0,1]$ be a Liouville density.

$$\rho_1 := \int_{\mathcal{P}(\mathcal{H}_2)} Seg^* \rho \; d\mu_2 \qquad \rho_2 := \int_{\mathcal{P}(\mathcal{H}_1)} Seg^* \rho \; d\mu_1$$

$$E(\rho) = \int_{\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)} |Seg^* \rho(p_1, p_2) - \rho_1(p_1)\rho_2(p_2)|^2 d\mu_1(p_1) d\mu_2(p_2)$$

Quantum control

Controlled *n*-level quantum system

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t)\right]|\psi(t)\rangle \quad , \quad |\psi(0)\rangle = |\psi_0\rangle$$

Quantum control

Controlled n-level quantum system

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t) \right] |\psi(t)\rangle \quad , \quad |\psi(0)\rangle = |\psi_0\rangle$$

$$i\hbar \frac{d}{dt} U(t) = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t) \right] U(t) \quad , \quad U(0) = \mathbb{I}$$

Quantum control

Controlled n-level quantum system

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t)\right]|\psi(t)\rangle \quad , \quad |\psi(0)\rangle = |\psi_0\rangle$$

$$i\hbar\frac{d}{dt}U(t) = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t)\right]U(t) \quad , \quad U(0) = \mathbb{I}$$

Complete quantum controllability

The n-level system is **completely controllable** if for any unitary operator $U_f \in U(n)$ there exist controls $u_1, \ldots, u_m$ and $T > 0$ such that $U(T) = U_f$.

Quantum control

Geometric Hamiltonian formulation

$$\dot{p}(t) = X_0(p(t)) + \sum_{i=1}^{m} X_i(p(t))u_i(t) \quad , \quad p(0) = p_0$$

X_i are the Hamiltonian fields on $\mathcal{P}(\mathcal{H})$ defined by the classical-like Hamiltonians obtained with our prescription.

Quantum control

Geometric Hamiltonian formulation

$$\dot{p}(t) = X_0(p(t)) + \sum_{i=1}^{m} X_i(p(t))u_i(t) \quad , \quad p(0) = p_0$$

X_i are the Hamiltonian fields on $\mathcal{P}(\mathcal{H})$ defined by the classical-like Hamiltonians obtained with our prescription.

Accessibility algebra: Lie algebra $\mathcal{C}$ generated by $\{X_0, ..., X_m\}$.
Rank condition: $\dim span\{X(p)|X \in \mathcal{C}\} = \dim \mathcal{P}(\mathcal{H}) \; \forall p \in \mathcal{P}(\mathcal{H})$

Quantum control

Geometric Hamiltonian formulation

$$\dot{p}(t) = X_0(p(t)) + \sum_{i=1}^{m} X_i(p(t))u_i(t) \quad , \quad p(0) = p_0$$

X_i are the Hamiltonian fields on $\mathcal{P}(\mathcal{H})$ defined by the classical-like Hamiltonians obtained with our prescription.

Accessibility algebra: Lie algebra $\mathcal{C}$ generated by $\{X_0, ..., X_m\}$.
Rank condition: $\dim span\{X(p)|X \in \mathcal{C}\} = \dim \mathcal{P}(\mathcal{H})\ \forall p \in \mathcal{P}(\mathcal{H})$

Theorem (D.P.2015)

Consider a quantum system described by a finite dimensional bilinear model, the following facts are equivalent:

1. *The system is completely controllable;*
2. *Rank condition is satisfied within geometric formulation;*
3. $\mathcal{C}$ *is the Lie algebra of g-Killing vector fields on* $\mathcal{P}(\mathcal{H})$.

Thank you for your attention!