On Asymptotic Expansions for Spin Boson Models

Markus Lange

Friedrich-Schiller-University Jena

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Joint work with G. Bräunlich, D. Hasler

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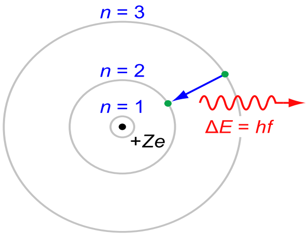


Source: Wikipedia

1. Model, Introduction

We introduce the symmetric Fock space

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}_n,$$

where the so called *n*-particle subspace is defined by

$$\begin{aligned}\mathcal{F}_n \quad := \quad & L_s^2((\mathbb{R}^3)^n) := \{\psi \in L^2((\mathbb{R}^3)^n) : \psi(k_1, ..., k_n) = \psi(k_{\pi(1)}, ..., k_{\pi(n)}) \\ & \qquad \forall \text{ permutations } \pi \text{ of } \{1, ..., n\}\}.\end{aligned}$$

We define the so called vacuum vector $\Omega = (1, 0, 0, \cdots)$.
Free field Hamiltonian is defined by

$$\begin{aligned}& H_f : D(H_f) \subset \mathcal{F} \to \mathcal{F} \\ & (H_f\psi)_n(k_1, ..., k_n) := (|k_1| + |k_2| + \cdots + |k_n|)\psi_n(k_1, ..., k_n).\end{aligned}$$

Spectrum is $\sigma(H_f) = [0, \infty)$.

We define the atomic Hamiltonian $H_{\rm at}$ as a selfadjoint operator on a finite dimensional Hilbert space $\mathcal{H}_{\rm at}$. The total Hilbertspace is defined by

$$\mathcal{H} := \mathcal{H}_{\rm at} \otimes \mathcal{F} \simeq \bigoplus_{n=0}^{\infty} L_s^2((\mathbb{R}^3)^n; \mathcal{H}_{\rm at}).$$

The Hamiltonian of the interacting system is

$$H(\lambda) = H_{\rm at} \otimes 1_{\mathcal{F}} + 1_{\mathcal{H}_{\rm at}} \otimes H_f + \lambda V.$$

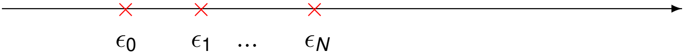
where $\lambda \in \mathbb{R}$ is the coupling constant and the interaction is given by

$$V := a(G) + a^*(G)$$
$$(a(G)\psi)_n(k_1, ..., k_n) := \sqrt{n+1} \int G(k)^* \psi_{n+1}(k, k_1, ..., k_n) dk,$$
$$G \in L^2(\mathbb{R}^3, \mathcal{L}(\mathcal{H}_{\rm at}), (|k|^{-2} + 1)dk).$$

Lemma. (Kato-Rellich.) For all $\lambda \in \mathbb{R}$ the Hamiltonian $H(\lambda)$ is selfadjoint on the natural domain of $H(0)$.

Spectral Properties

- Spectrum of H_{at}: $\{\epsilon_0, ..., \epsilon_N\}$



- Spectrum of $H(0) = H_{\text{at}} \otimes 1 + 1 \otimes H_f$: $[E(0), \infty)$

- Spectrum of $H(\lambda)$ for $\lambda \neq 0$: $[E(\lambda), \infty)$ where $E(\lambda) := \inf \sigma(H(\lambda))$.

Theorem (Bach-Fröhlich-Sigal '98, Griesemer–Lieb–Loss '00, Gerard '00)

The number $E(\lambda)$ ist an eigenvalue of $H(\lambda)$, i.e., there exits a nonzero $\psi(\lambda) \in \mathcal{H}$ such that $H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda)$.

2. Expansions

Question: How does the ground state and the ground state energy depend on the coupling constant λ? Or more general, how do they depend on other parameters of H_{at} or V?

• The answer is useful for scattering theory, adiabatic theory, ...

• Since there is no gap in the spectrum, this question is mathematically difficult to answer.

Some results addressing this question:

- Expansions in the fine structure constant for non-relativistic qed

  UV Cutoff $\sim$ energy electron: Hainzl-Seiringer '02, Barbaroux-Chen-Vougalter-Vougalter '08,....

  UV Cutoff $\sim$ binding energy: Bach-Fröhlich-Pizzo '06, Hasler-Herbst '08, ...

- Translation invariant models. Regularity Properties of the ground state energy as a function of total momentum.

  C^0-,C^2-properties: Fröhlich '73, Chen '08, Pizzo '03, Bach-Fröhlich-Pizzo 06,...

  Analyticity: Abdessalam-Hasler '12, Faupin-Fröhlich-Schnubel '14,...

- Analyticity results.

  Non-relativistic qed: Hasler-Herbst '11

  Spin boson type models: Griesemer-Hasler'09, Hasler-Herbst'11

We shall make the following assumption

Hypothesis 1. There exists a positive λ_0 such that for all $\lambda \in [0, \lambda_0]$ the number $E(\lambda)$ is an eigenvalue of $H(\lambda)$ with eigenvector $\psi(\lambda) \in \mathcal{H}$.

Hypothesis 1 can be verified for our model. (Gerard '00)

Theorem 1

Suppose that Hypothesis 1 holds and that the ground state of H_{at} is nondegenerate. Then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{\lambda \downarrow 0} \lambda^{-n} \left(E(\lambda) - \sum_{k=0}^{n} E_k \lambda^k \right) = 0.$$

The sequence is determined by (1)–(3)*, below.*

Let $\varphi_{\rm at}$ be the normalized ground state and E_0 the ground state energy of $H_{\rm at}$. We define $\psi_0 := \varphi_{\rm at} \otimes \Omega$, $P_0 := |\psi_0\rangle\langle\psi_0|$ and $\bar{P}_0 := 1 - P_0$.

Theorem 2

Suppose that Hypothesis 1 holds and that the ground state of $H_{\rm at}$ is nondegenerate. Then there exists a unique sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$E_1 = \langle \psi_0, V\psi_0 \rangle, \tag{1}$$

$$E_n = \lim_{\eta \downarrow 0} E_n(\eta), \quad n \geq 2 \tag{2}$$

where

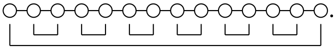
$$E_n(\eta) := \sum_{k=2}^{n} \sum_{\substack{j_1+\cdots+j_k=n \\ j_s \geq 1}} \langle \psi_0, (\delta_{1 j_1} V - E_{j_1}) \prod_{s=2}^{k} \left\{ \frac{\bar{P}_0}{E_0 - \eta - H_0} (\delta_{1 j_s} V - E_{j_s}) \right\} \psi_0 \rangle. \tag{3}$$

Starting to calculate the right hand side of (3), we obtain, using a generalized form of Wicks Theorem

$$
\begin{aligned}
&E_{2m}(\eta)(-1)^{2m-1} \\
&= \int dk_1 ... dk_m \langle \varphi_{\mathrm{at}}, G^*(k_1) \frac{P_{\mathrm{at}}}{|k_1| + \eta} G^*(k_2) \frac{P_{\mathrm{at}}}{|k_1| + |k_2| + \eta} G(k_2) \frac{P_{\mathrm{at}}}{|k_1| + \eta} \\
&\qquad \cdots G^*(k_m) \frac{P_{\mathrm{at}}}{|k_1| + |k_m| + \eta} G(k_m) \frac{P_{\mathrm{at}}}{|k_1| + \eta} G(k_1) \varphi_{\mathrm{at}} \rangle + \cdots ,
\end{aligned}
$$

where P_{at} denotes the projection onto φ_{at}. If $\eta \downarrow 0$ the integral over k_1 can become divergent for large m.

Illustration.



The finiteness as $\eta \downarrow 0$ can be restored using cancellations due to energy subtractions.

$$
\begin{aligned}
&- \int dk_2 P_{\mathrm{at}} G^*(k_2) \frac{1}{|k_1| + |k_2| + \eta} G(k_2) P_{\mathrm{at}} - E_2 P_{\mathrm{at}} \\
&\quad = - \left(\int dk_2 P_{\mathrm{at}} G^*(k_2) \left(\frac{1}{|k_1| + |k_2| + \eta} - \frac{1}{|k_2|} \right) G(k_2) P_{\mathrm{at}} \right) \\
&\quad = (|k_1| + \eta) \int dk_2 P_{\mathrm{at}} G^*(k_2) \frac{1}{(|k_1| + |k_2| + \eta)|k_2|} G(k_2) P_{\mathrm{at}}.
\end{aligned}
$$

This improves the singularity in k_1!

Really short idea of the proof

• Interpret every term of the Wick ordered product as a contraction of some 'graph-function'. (Illustration)

• Show that there are energy subtraction up to arbitrary order

• Show that these energy subtraction yield finite expressions at every order

Summary, Outlook

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• We considered asymptotic expansions for the energy for a class of models of quantum field theory.

Outlook

• Estimate the growth of the coefficients E_n.

• Consider degenerate situations.

• Generalize expansions to non-relativistic QED.

Thank you for you attention!