Essential Spectrum of Singular Matrix Differential Operators

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(based on a joint work with P. Siegl and C. Tretter)

February 12, 2016

- **The operator matrix:** In $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$, consider

$$\mathcal{A} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}t} p \frac{\mathrm{d}}{\mathrm{d}t} + q & -\frac{\mathrm{d}}{\mathrm{d}t} \overline{b} + \overline{c} \\ b \frac{\mathrm{d}}{\mathrm{d}t} + c & d \end{pmatrix}$$

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- Singular MDOs and related spectral problems are ubiquitous e.g. in stability problems of magnetohydrodynamics, fluid dynamics and astrophysics.

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Example. Consider, for $m \in \mathbb{N}$,

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- $\sigma_{\mathrm{ess}}(\mathcal{A}_\infty)$? Maybe $\sigma_{\mathrm{ess}}(\mathcal{A}_\infty) = \mathrm{cl}\Big\{\bigcup_{m\in\mathbb{N}} \sigma_{\mathrm{ess}}(\mathcal{A}_m)\Big\} = [-1,0]$?

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Descloux - Geymonat conjecture '80

the appearance of the singular part in a model of magnetohydrodynamics

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- the regular and singular parts are always adjoined to each other?

Main tools

- **The first Schur complement** associated to

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is given by

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- The Schur complement is given by, for $\lambda \in \mathbb{C} \setminus \sigma(d)$,

$$S(\lambda) =: -\pi(\cdot, \lambda)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \rho(\cdot, \lambda)\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \kappa(\cdot, \lambda),$$

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Recall

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- $\det(\mathcal{A} - \lambda) = -\big(\Delta(t) - \lambda\big)\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ + "lower order terms", where

$$\Delta(t) := d(t) - \frac{|b(t)|^2}{p(t)}, \quad t \in [0, \infty).$$

- The Schur complement is given by, for $\lambda \in \mathbb{C} \setminus \sigma(d)$,

$$S(\lambda) =: -\pi(\cdot, \lambda)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \rho(\cdot, \lambda)\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \kappa(\cdot, \lambda),$$

where

$$\pi(\cdot, \lambda) := p - \frac{|b^2|}{d - \lambda}, \quad \rho(\cdot, \lambda) := -\frac{2\,\mathrm{Im}(b\overline{c})}{d - \lambda} + \mathrm{i}\frac{\partial}{\partial t}\pi(\cdot, \lambda),$$

$$\kappa(\cdot, \lambda) := q - \lambda - \frac{|c|^2}{d - \lambda} + \frac{\partial}{\partial t}\mathrm{Re}\Big(\frac{\overline{b}c}{d - \lambda}\Big).$$

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- earlier works concern special cases of classification in terms of $\pi_0(\lambda), \pi_1(\lambda)$:
  - in Kurasov, Lelyavin, Naboko (2008): $\pi_0(\lambda) \neq 0$ or $\pi_0(\lambda) = 0, \pi_1(\lambda) \neq 0$.
  - in Kurasov, Naboko (2003): $\pi_0(\lambda) = \pi_1(\lambda) = 0$;
  - in Möller (2004): $\pi_0(\lambda) \neq 0$;
  - in Mennicken, Naboko, Tretter (2002): $\pi_0(\lambda) = \pi_1(\lambda) = 0$.

Application to a spectral problem for symmetric stellar equilibrium models

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In $L^2(0,R) \oplus L^2(0,R)$, consider

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with coefficient functions

$$p_1 := \frac{\Gamma_1 p}{\varrho}, \quad p_2 := c\frac{\Gamma_1 p}{t\varrho}, \quad p_3 := c^2 \frac{\Gamma_1 p}{t^2 \varrho}, \quad \ldots$$

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- p, q are related to Lane-Emden equation:

$$\theta''(t) + \frac{2}{t}\theta'(t) = -\frac{1}{\alpha^2}\theta(t)^n, \quad t \in (0,\infty)$$

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Thanks for your attention!