Schrödinger equation with nonlinear defect as effective model

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Preliminaries

Joint work with C.Cacciapuoti, D.Noja, A.Teta

Preprint arXiv:1511.06731

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- The NLS is an ubiquitous equation describing several nonlinear phenomena
- There are simplified models where the nonlinearity is concentrated at point
- Widely used in physics for diffraction of electrons from a thin layer, analysis of nonlinear resonant tunneling, models of soliton bifurcation and so on. At a formal level, they correspond to the equation

$$i\frac{d}{dt}\psi(t) = -\Delta\psi(t) + \gamma\,\delta_0|\psi(t)|^{2\mu}\psi(t)$$

Preliminaries

The correct mathematical formulation of the previous equation is given by:

$$\psi(t,\mathbf{x}) = (U(t)\psi_0)(\mathbf{x}) + i\int_0^t U(t-s,\mathbf{x})q(s)\mathrm{d}s \quad U(t,\mathbf{x}) = \frac{\exp(i\frac{x^2}{4t})}{(4\pi i t)^{3/2}}$$

$$q(t) + 4\sqrt{\pi i}\gamma\int_0^t \frac{|q(s)|^{2\mu}q(s)}{\sqrt{t-s}}ds = 4\sqrt{\pi i}\int_0^t \frac{(U(s)\psi_0)(0)}{\sqrt{t-s}}\mathrm{d}s$$

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Here we discuss the range of application of this equation: how it can be derived as an effective equation from a more fondamental one. We focus on the concentration of the interaction in a single point.

The one dimensional case and the three dimensional case are quite different.

Linear case

Delta-interaction in dimension three

$$H_{\alpha} = -\Delta + \alpha\delta_0$$

$$\mathcal{D}(H_{\alpha}) = \left\{\psi = \phi + \frac{q}{4\pi|\mathbf{x}|};\ \phi \in \dot{H}^2(\mathbb{R}^3)\,;\ q \in \mathbb{C};\ \phi(\mathbf{0}) = \alpha q\right\}$$

and

$$H_{\alpha}\psi = -\Delta\phi \qquad \mathbf{x} \neq \mathbf{0}$$

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The evolution can be represented by

$$\psi(t,\mathbf{x}) = (U(t)\psi_0)(\mathbf{x}) + i\int_0^t U(t-s,\mathbf{x})q(s)\mathrm{d}s$$

$$q(t) + 4\sqrt{\pi i}\int_0^t \frac{\alpha q(s)}{\sqrt{t-s}}\mathrm{d}s = 4\sqrt{\pi i}\int_0^t \frac{(U(s)\psi_0)(0)}{\sqrt{t-s}}\mathrm{d}s$$

Nonlinear case

The nonlinear model is obtained by introducing a nonlinear coupling $\alpha \to \alpha(\psi) = \gamma |q|^{2\mu}$

$$\psi(t, \mathbf{x}) = (U(t)\psi_0)(\mathbf{x}) + i \int_0^t U(t-s, \mathbf{x}) q(s) ds$$

$$q(t) + 4\sqrt{\pi} i \gamma \int_0^t \frac{|q(s)|^{2\mu} q(s)}{\sqrt{t-s}} ds = 4\sqrt{\pi} i \int_0^t \frac{(U(s)\psi_0)(0)}{\sqrt{t-s}} ds$$

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Define

$$\mathcal{D} = \left\{\psi = \phi + \frac{q}{4\pi|\mathbf{x}|};\ \phi \in \dot{H}^2(\mathbb{R}^3);\ q \in \mathbb{C};\ \phi(\mathbf{0}) = \gamma|q|^{2\mu}q\right\}$$

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Conserved quantities:

$$\mathcal{E}(\psi) = \int dx\,|\nabla\phi|^2 + \frac{\gamma}{\mu+1}|q|^{2\mu+2} \qquad \mathcal{M}[\psi] = \|\psi\|_{L^2}^2$$

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Theorem

Let $\psi_0 \in \mathcal{D}$, if $\gamma > 0$ and $\forall \mu > 0$ or if $\gamma < 0$ and $0 < \mu < 1$ then there is a a global solution $\psi \in C([0, T], \mathcal{D}) \cap C^1([0.T]. L^2(\mathbb{R}^3))$. Moreover energy and mass are conserved along the solutions.

Approximating problem in the linear case

The approximation of a three dimensional delta-interaction is more delicate than the one dimensional case.

$$-\Delta + \frac{1}{\varepsilon^3} V\left(\frac{x}{\varepsilon}\right) \not\to -\Delta + \alpha\delta_0$$

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More subtle phenomena are involved: resonant potential for local approximation, renormalization for non local approximation.

$$H_\varepsilon = -\Delta - \beta^\varepsilon |\rho^\varepsilon\rangle\langle\rho^\varepsilon| \qquad \rho^\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^3}\rho\left(\frac{\mathbf{x}}{\varepsilon}\right)$$

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$$\frac{1}{\beta^\varepsilon} = \frac{\ell}{\varepsilon} + \alpha \qquad \ell = \int d\mathbf{k}\, \frac{\hat{\rho}^2(\mathbf{k})}{|\mathbf{k}|^2}$$

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$$H_\varepsilon = -\Delta + \frac{\varepsilon}{\ell}\left(-1 + \alpha\frac{\varepsilon}{\ell}\right) |\rho^\varepsilon\rangle\langle\rho^\varepsilon|$$

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The linear case suggest as non local approximation:

$$i\frac{d}{dt}\psi^{\varepsilon}(t) = -\Delta\psi^{\varepsilon} + \left(-1 + \gamma\frac{\varepsilon^{2\mu+1}|(\rho^{\varepsilon},\psi^{\varepsilon}(t))|^{2\mu}}{\ell^{2\mu+1}}\right)\frac{\varepsilon}{\ell}(\rho^{\varepsilon},\psi^{\varepsilon}(t))\rho^{\varepsilon}$$

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For finite ε this is a well behaved NLS on $H^2(\mathbb{R}^3)$ while the solutions of the limit equation belongs to $\mathcal{D}$ which has trivial intersection with $H^2(\mathbb{R}^3)$.

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Therefore if we want to work with H^2-solutions of the approximating problem, we need a smooth initial datum which approximate the initial datum on $\mathcal{D}$:

$$\psi_0 = \phi_0 + \frac{q_0}{4\pi|\mathbf{x}|} \qquad \psi_0^{\varepsilon} = \phi_0 + \frac{q_0}{4\pi}(\rho^{\varepsilon} * \frac{1}{|\cdot|})(\mathbf{x})$$

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Moreover it is natural to prove the convergence of ψ^ε to ψ in the L^2-norm.

Main theorem

Theorem

Let $\psi_0 \in \mathcal{D}$ *and assume* $\gamma > 0$ *or* $\gamma < 0$ *and* $0 < \mu < 1$. *Let* $\psi(t)$ *be the solution of the limit problem and let* $\psi^\varepsilon(t)$ *the solution of the approximating problem with the initial datum as discussed before. Then for any* $T > 0$ *we have*

$$\sup_{t \in [0,T]} \|\psi^\varepsilon(t) - \psi(t)\|_{L^2} \leq c\, \varepsilon^\delta \qquad 0 \leq \delta < 1/4$$

Ideas from the proof

We have to reconstruct the structure of the limit in the approximating problem.

$$q^\varepsilon(t) = \frac{\varepsilon}{\ell}(\rho^\varepsilon, \psi^\varepsilon(t)) \quad \phi^\varepsilon(t) = \psi^\varepsilon(t) - q^\varepsilon(t)\rho^\varepsilon * \frac{1}{4\pi|\cdot|}$$

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The approximating problem takes the form:

$$\psi^{\varepsilon}(t, \mathbf{x}) = (U(t)\psi_0^{\varepsilon})(\mathbf{x}) - i\int_0^t \mathrm{d}s\, U(t-s)\rho^{\varepsilon}(\mathbf{x}) \left(-1 + \gamma\frac{\varepsilon}{\ell}|q^{\varepsilon}(s)|^{2\mu}\right) q^{\varepsilon}(s)$$

Ideas from the proof

Convergence is reduced to the convergence of initial datum and convergence of charge in a suitable topology

$$\sup_{t\in[0,T]} \|\psi^\varepsilon(t)-\psi(t)\|_{L^2} \leq c \left(\|\psi_0^\varepsilon-\psi_0\|_{L^2} + \|I^{1/2}(q-q^\varepsilon)\|_{L^\infty(0,T)} + \varepsilon^{1/4} \right)$$

$$I^{1/2}f(t) = \int_0^t \mathrm{d}s\, \frac{f(s)}{\sqrt{t-s}}$$

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$$I^{1/2}f(t) = \int_0^t \mathrm{d}s \, \frac{f(s)}{\sqrt{t-s}}$$

An equation for $I^{1/2}q$ can be easily derived and we have

$$I^{1/2}q(t) + 4\pi\sqrt{\pi i}\gamma \int_0^t \mathrm{d}s \, |q(s)|^{2\mu} q(s) = 4\pi\sqrt{\pi i}\gamma \int_0^t \mathrm{d}s \, (U(s)\psi_0)(\mathbf{0})$$

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On the contrary for $I^{1/2}q^\varepsilon$ we do not obtain a closed equation

$$I^{1/2}q^\varepsilon(t) + 4\pi\sqrt{\pi i}\gamma \int_0^t \mathrm{d}s\, |q^\varepsilon(s)|^{2\mu} q^\varepsilon(s) = 4\pi\sqrt{\pi i}\gamma \int_0^t \mathrm{d}s\, (U(s)\psi_0)(\mathbf{0}) + Y^\varepsilon(t)$$

Ideas from the proof

The source term has the following expression

$$Y^\varepsilon(t) = \sum_{j=1}^{4} Y_j^\varepsilon(t),$$

$$Y_1^\varepsilon(t) = -(4\pi)^2\sqrt{\pi i}\int_0^t \mathrm{d}\tau\, q^\varepsilon(\tau)\int_0^\infty \mathrm{d}k\,\left((\hat\rho(\varepsilon k))^2 - (\hat\rho(0))^2\right) e^{-ik^2(t-\tau)},$$

$$Y_2^\varepsilon(t) = (4\pi)^2\sqrt{\pi i}\gamma\frac{\varepsilon}{\ell}\int_0^t \mathrm{d}\tau |q^\varepsilon(\tau)|^{2\mu} q^\varepsilon(\tau)\int_0^\infty \mathrm{d}k\,\left((\hat\rho(\varepsilon k))^2 - (\hat\rho(0))^2\right)\, e^{-ik^2(t-\tau)},$$

$$Y_3^\varepsilon(t) = \gamma\frac{\varepsilon}{\ell}\int_0^t \mathrm{d}\tau\,\frac{|q^\varepsilon(\tau)|^{2\mu} q^\varepsilon(\tau)}{\sqrt{t-\tau}},$$

$$Y_4^\varepsilon(t) = 4\pi\sqrt{\pi i}\left(\int_0^t \mathrm{d}s\,(\rho^\varepsilon, U(s)\psi_0^\varepsilon) - \int_0^t \mathrm{d}s (U(s)\psi_0)(\mathbf{0})\right).$$

Ideas from the proof

Starting from

$$I^{1/2}(q^{\varepsilon}-q)(t)+4\pi\sqrt{\pi i}\,\gamma\int_0^t ds(|q^{\varepsilon}(s)|^{2\mu}q^{\varepsilon}(s)-|q(s)|^{2\mu}q(s))=Y^{\varepsilon}(t).$$

to prove the convergence of $I^{1/2}q^{\varepsilon}$ to $I^{1/2}q$ it s sufficient to prove the following estimates:

$$\|Y^{\varepsilon}\|_{L^{\infty}(0,T)}\leq C\varepsilon^{1/2}$$

$$\|D^{1/2}Y^{\varepsilon}(t)\|_{L^{1}(0,T)}\leq C\varepsilon^{\delta}$$

Ideas from the proof

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$$I^{1/2}(q^\varepsilon - q)(t) + 4\pi\sqrt{\pi i}\,\gamma\int_0^t ds(|q^\varepsilon(s)|^{2\mu}q^\varepsilon(s) - |q(s)|^{2\mu}q(s)) = Y^\varepsilon(t).$$

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Some a priori estimates are used

$$\|q^\varepsilon\|_{L^\infty} \le c \qquad \sup_t \|\nabla\phi^\varepsilon(t)\|_{L^2} \le c$$

$$\|\dot{q}^\varepsilon\|_{L^\infty} \le c\varepsilon^{-3/2} \qquad \|D^{1/2}q^\varepsilon\|_{L^1} \le c\varepsilon^{-1/2+\delta}$$

Notice that the first couple holds only on the same range of parameters where the limit problem has a global solution.

Perspectives

Some perspectives:

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- 3d nonlocal on form domain: space-time norm

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- 3d local approximation with resonant potential

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- 3d nonlocal on form domain: space-time norm
- 3d local approximation with resonant potential
- 1d local approximation with singular scaling of resonant potential

1d linear case

Self-adjoint operator H_α

$$H_\alpha = -\frac{d^2}{dx^2} + \alpha\delta_0 \qquad \alpha \in \mathbb{R}$$

$$\mathcal{D}(H_\alpha) = \left\{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}),\ \psi'(0+) - \psi'(0-) = \alpha\psi(0)\right\}$$

$$H_\alpha\psi = -\psi'' \qquad \forall x \neq 0$$

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$$H_\alpha\psi \ = -\psi'' \qquad \forall x \neq 0$$

Representation for the unitary group generated by H_α

$$\begin{cases} i\dfrac{d}{dt}\psi(t) = H_\alpha\psi(t) \\ \psi(0) = \psi_0 \end{cases} \qquad \psi_0 \in H^1$$

$$\psi(t,x) = (U(t) * \psi_0)(x) - i\int_0^t \alpha U(t-s,x)\psi(s,0)ds$$

$$U(t,x) = \frac{1}{\sqrt{4\pi i t}} e^{i\frac{x^2}{4t}}$$

1d nonlinear case

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$$\mathcal{E}(\psi(t)) = \int dx|\psi'(t,x)|^2 + \frac{\gamma}{\mu+1}|\psi(t,0)|^{2\mu+2}$$

Theorem

This equation has a global solution for $\psi_0 \in H^1(\mathbb{R})$ if $\gamma > 0$ and $\forall\mu > 0$ or if $\gamma < 0$ and $0 < \mu < 1$. Moreover energy is conserved along the solutions.

Approximating problem

We know that in one dimension

$$-\frac{d}{dx^2} + \frac{1}{\varepsilon} V\left(\frac{x}{\varepsilon}\right) \longrightarrow -\frac{d}{dx^2} + \alpha\delta_0 \quad \alpha = \int V(x)$$

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For $\psi_0 \in H^1(\mathbb{R})$ we define an approximating problem

$$\psi^\varepsilon(t,x) = U(t)\psi_0(x) - i \int_0^t ds \int dy \, U(t-s, x-y) \, \frac{1}{\epsilon} V\left(\frac{y}{\epsilon}\right) \, |\psi^\varepsilon(s,y)|^{2\mu} \psi^\varepsilon(s,y)$$

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$$i\frac{d}{dt}\psi^\varepsilon(t,x)=-\psi^{\varepsilon\prime\prime}(t,x)+\frac{1}{\epsilon}V\left(\frac{x}{\epsilon}\right)\,|\psi^\varepsilon(t,x)|^{2\mu}\psi^\varepsilon(t,x)$$

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$$i\frac{d}{dt}\psi^\varepsilon(t,x) = -\psi^{\varepsilon\prime\prime}(t,x) + \frac{1}{\epsilon} V\left(\frac{x}{\epsilon}\right) |\psi^\varepsilon(t,x)|^{2\mu}\psi^\varepsilon(t,x)$$

Also for the approximating problem energy is conserved

$$\mathcal{E}^\varepsilon(\psi^\varepsilon(t)) = \int dx |\psi^{\varepsilon\prime}(t,x)|^2 + \frac{1}{\mu+1}\int_{\mathbb{R}} dx\, \frac{1}{\varepsilon} V\left(\frac{x}{\varepsilon}\right) |\psi^\varepsilon(t,x)|^{2\mu+2}$$

Main theorem in one dimension

The main result is the following

Theorem

Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ *and* $\gamma = \int V dx$*. Let* $V \geq 0$ *or* $\mu \in (0,1)$ *then for any* $T > 0$ *we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|\psi^\varepsilon(t) - \psi(t)\|_{H^1} = 0$$

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- under the above hypothesis both the approximating problem and the limit one have global solutions
- notice that the limit problem in the focusing case is global only in the sub cubical case

Ideas from the proof

The following a priori estimate is crucial

A priori estimate

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Notice that it implies a uniform bound on the L^∞ norm

Ideas from the proof

Since the limit evolution has the form

$$\psi(t,x) = (U(t) * \psi_0)(x) - i\gamma \int_0^t U(t-s,x)|\psi(s,0)|^{2\mu}\psi(s,0)ds$$

the first step is the convergence of $\psi^\varepsilon(t,0)$

Convergence in the defect

Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ and let $V \geq 0$ or $\mu \in (0,1)$ then for any $T > 0$ and $0 < \delta < 1/2$ we have

$$\sup_{t \in (0,T)} |\psi^\varepsilon(t,0) - \psi(t,0)| \leq c\, \varepsilon^\delta$$

Ideas from the proof

As intermediate step we prove convergence in $L^2(\mathbb{R})$

L^2-convergence

Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ and let $V \geq 0$ or $\mu \in (0,1)$ then for any $T > 0$ and $0 < \delta < 1/2$ we have

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Take $V \in L^1(\mathbb{R}, \langle x \rangle\, dx) \cap L^\infty(\mathbb{R})$ and let $V \geq 0$ or $\mu \in (0,1)$ then for any $T > 0$ and $0 < \delta < 1/2$ we have

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The convergence is strengthened to H^1 by a soft argument but we lose the rate.

Remarks

Some remarks

- the proof holds for N defects not just one
- nonlocal approximations are also possible
- notice that we assume the positivity of V not of $\gamma = \int V$
- we could soften the hypothesis on V