Eigenvalue Bounds for Dirac and Fractional Schrödinger Operators with Complex Potentials

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Mathematical Challenges in Quantum Mechanics
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Outline

1. Motivation
   - Lieb-Thirring Inequalities (S.A.)
   - Lieb-Thirring Inequalities (N.S.A.)
2. New Results
   - Dirac and Fractional Schrödinger Operators
   - Method of Proof

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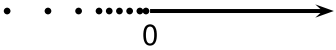
Lieb-Thirring Inequalities (S.A.)

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$. Then $H_0^* = H_0$ and $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [0, \infty)$.
- $H = H_0 + V$, where $V \in L^{d/2+\gamma}(\mathbb{R}^d; \mathbb{R})$.

Lieb-Thirring and CLR inequalities

$$\sum_{E \in \sigma_{\mathrm{d}}(H)} |E|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_-(x)^{d/2+\gamma} dx, \quad \begin{cases} \gamma \geq 0 & \text{if } d \geq 3, \\ \gamma > 0 & \text{if } d = 2, \\ \gamma \geq 1/2 & \text{if } d = 1. \end{cases}$$



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Preliminaries: Non-Selfadjoint Operators

X a Banach space, T a closed operator in X.

Definition

- $\rho(T) := \{z \in \mathbb{C} : \; T - z \text{ is bijective}\}$, $\sigma(T) := \mathbb{C} \setminus \rho(T)$.
- $\sigma_{\mathrm{d}}(T) := \{z \in \mathbb{C} : \; z \text{ isolated e.v. of finite algebraic mult.}\}$.
- $\sigma_{\mathrm{ess}}(T) := \{z \in \mathbb{C} : \; T - z \text{ is not Fredholm}\}$.

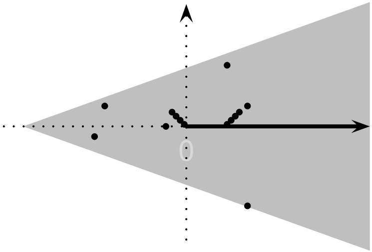
Fact 1: $(T-z)^{-1} - (S-z)^{-1}$ compact $\implies \sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(S)$.

Fact 2: If each connected component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$ contains a point in $\rho(T)$, then

$$\sigma(T) = \sigma_{\mathrm{d}}(T) \dot{\cup} \sigma_{\mathrm{ess}}(T).$$

N.S.A. Schrödinger Operators: Single Eigenvalues

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$.
- $H = H_0 + V$, where $V \in L^{d/2+\gamma}(\mathbb{R}^d; \mathbb{C})$.
- Assume $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of H.



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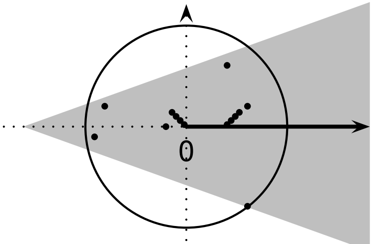

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Theorem (Abramov, Aslanyan, Davies 2001 [1])

Assume $d = 1$. Then

$$|z|^{1/2} \leq \frac{1}{2} \int_{\mathbb{R}} |V(x)| dx.$$

Proof.

$$1 \leq \| V^{1/2} R_0(z) |V|^{1/2} \|^2 \leq \int \int \frac{|V(x)| e^{-2\Im \sqrt{z}|x-y|} |V(y)|}{4|z|} \leq \frac{\|V\|_1^2}{4|z|}.$$

□

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Conjecture (Laptev, Safronov 2009 [10])

Assume $d \geq 1$ and $0 < \gamma \leq d/2$. Then

$$|z|^\gamma \leq C_{d,p} \int_{\mathbb{R}} |V(x)|^{d/2+\gamma} dx.$$

- Case $0 < \gamma \leq 1/2$ proved by R. Frank [6].
  Proof relies on uniform Sobolev inequality of C. Kenig, A. Ruiz, C. Sogge [9]:

$$\|R_0(z)\|_{L^p \to L^{p'}} \leq C|z|^{d(1/p-1/2)-1}, \qquad \frac{2d}{d+2} \leq p \leq \frac{2(d+1)}{d+3}.$$

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Conjecture (Laptev, Safronov 2009 [10])

Assume $d \geq 1$ and $d/2 \leq p \leq d$. Then

$$|z|^{p-d/2} \leq C_{d,p} \int_{\mathbb{R}} |V(x)|^p dx.$$

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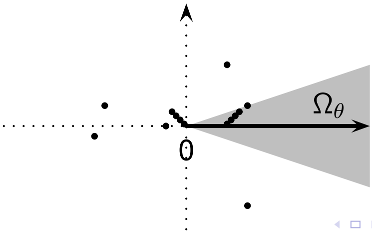
N.S.A. Schrödinger Operators: Sums of Eigenvalues

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$.
- $H = H_0 + V$, where $V \in L^p(\mathbb{R}^d; \mathbb{C})$.

Theorem (Frank, Laptev, Lieb, Seiringer 2006 [7])

For $\theta \in (0, \pi/2)$ and $p \geq d/2 + 1$:

$$\sum_{z \in \sigma_{\mathrm{d}}(H) \setminus \Omega_\theta} |z|^{p-d/2} \leq C_{d,p}(1 + 2/\tan(\theta))^p \|V\|_p^p.$$



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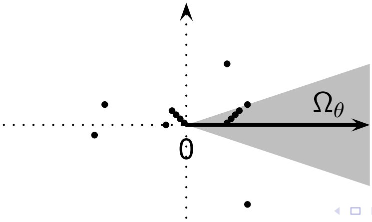
Theorem (Demuth, Hansmann, Katriel 2009 [2])

For $p \geq d/2 + 1$, $d \geq 0$, $\epsilon > 0$ and $\Re(V) \geq 0$:

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} \frac{dist(z, [0,\infty))^{p+\epsilon}}{|z|^{d/2}(1+|z|)^{2\epsilon}} \leq C_{d,p} \|V\|_p^p.$$



N.S.A. Schrödinger Operators: Sums of Eigenvalues

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- $H = H_0 + V$, where $V \in L^p(\mathbb{R}^d; \mathbb{C})$.

Conjecture (Demuth, Hansmann, Katriel [3])

For $p > d/2$:

$$\sum_{z \in \sigma_{\mathrm{d}}(H)} \frac{dist(z, [0,\infty)^p}{|z|^{d/2}} \leq C_{d,p} \|V\|_p^p.$$

- In part. $\sigma_{\mathrm{d}}(H) \ni z_n \to z^* \in (0,\infty) \implies (\Im z_n)_{n \in \mathbb{N}} \subset l^p(\mathbb{N})$.
- What is the lowest possible p?

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Dirac and Fractional Schrödinger Operators

Fractional Schrödinger Operator

- $H_0 = (m^2 - \Delta)^{s/2}$ in $L^2(\mathbb{R}^d)$, with $0 < s < d$ and $D(H_0) = H^s(\mathbb{R}^d)$. Then $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [m, \infty)$
- $H = H_0 + V$, where $V \in L^p(\mathbb{R}^d; \mathbb{C})$ or $V \in (L^p \cap L^q)(\mathbb{R}^d; \mathbb{C})$.

Dirac Operator

- $H_0 = \alpha \cdot D + m\beta$ in $L^2(\mathbb{R}^d, \mathbb{C}^N)$, w. $D(H_0) = H^1(\mathbb{R}^d, \mathbb{C}^N)$. Then $\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = (-\infty, m] \cup [m, \infty)$.
- $H = H_0 + V$, where $V \in (L^p \cap L^q)(\mathbb{R}^d; \mathbb{C}^{N\times N})$.
- We define $s = 1$ in this case.

- Here: $m \geq 0$.
- The range of admissible p, q will depend on d and s.

Main result

Assumptions:

- $$\Lambda_{\mathrm{crit}}(H_0) = \begin{cases} \{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\ \{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta. \end{cases}$$
- Assume $$\begin{cases} V \in L^{p \in [d/s, (d+1)/2]} & \text{if } s \geq 2d/(d+1), \\ V \in L^{d/s} \cap L^{(d+1)/2} & \text{if } s < 2d/(d+1). \end{cases}$$

Theorem (Bounds for single eigenvalues)

Assume $H_0 = (m^2 - \Delta)^{s/2}$ with $s \geq 2d/(d+1)$, $d \geq 2$. Then all complex eigenvalues of H lie in a compact neighborhood of $\Lambda_{\mathrm{crit}}(H_0)$. In particular, for $m = 0 \Rightarrow |z|^{p - \frac{d}{s}} \leq \|V\|_p^p$.

- In $d = 1$ the Theorem is true for Dirac, but not for $(m^2 - \Delta)^{1/2}$.

Main result

Assumptions:

- $$\Lambda_{\mathrm{crit}}(H_0) = \begin{cases} \{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\ \{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta. \end{cases}$$
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Theorem (Bound on distribution of eigenvalues)

Let $(z_n)_n \subset \sigma_{\mathrm{d}}(H)$ such that $z_n \to z^* \in \sigma(H_0) \setminus \Lambda_{\mathrm{crit}}(H_0)$. Then $(dist(z_n, \sigma(H_0)))_{n \in \mathbb{N}} \in l^1(\mathbb{N})$.

- The case $s = 2$ is due to Frank and Sabin [8].
- Dubuisson [4]–[5] proved $(dist(z_n, \sigma(H_0)))_{n \in \mathbb{N}} \in l^p(\mathbb{N})$ for larger p.

Main result

Assumptions:

- $$\Lambda_{\mathrm{crit}}(H_0) = \begin{cases} \{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\ \{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta. \end{cases}$$
- Assume $$\begin{cases} V \in L^{p \in [d/s, (d+1)/2]} & \text{if } s \geq 2d/(d+1), \\ V \in L^{d/s} \cap L^{(d+1)/2} & \text{if } s < 2d/(d+1). \end{cases}$$

Corollary

$$\#\{z \in \sigma_{\mathrm{d}} : |\Im z| \geq s\} \leq \frac{C}{s}.$$

- If $d \geq 2d/(d+1)$ and $\|V\|_{L^{d/s}} \ll 1$, then $\nexists$ complex spectrum; in fact, H is similar to H_0 ($|V|^{1/2}$ is Kato-smooth w.r.t H_0).

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Perturbation Determinant

- Assume $V^{1/2}R_0(z)|V|^{1/2} \in \mathfrak{S}^\alpha(L^2(\mathbb{R}^d))$.
- Regularized determinant

$$f : \rho(H_0) \to \mathbb{C}, \quad f(z) := \det_{\lceil\alpha\rceil}(I + V^{1/2}R_0(z)|V|^{1/2}),$$

where for $A \in \mathfrak{S}^n(L^2(\mathbb{R}^d))$:

$$\det_n(I+A) := \prod_k [1+\lambda_k(A)] \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} \lambda_k(A)^j\right).$$

- f is holomorphic, and

$$f(z) = 0 \longleftrightarrow z \in \sigma_{\mathrm{d}}(H).$$

- $\ln|f(z)| \leq C_\alpha \| V^{1/2}R_0(z)|V|^{1/2}\|_{\mathfrak{S}^\alpha}$.

Jensen's Identity

- $h : \mathbb{D} \to \mathbb{C}$ holomorphic, $h(0) = 1$.
- $N(h; s)$ number of zeros of h in $B(0, s)$.

Jensen's identity: $\forall r \in (0, 1)$

$$\int_0^r \frac{N(h; s)}{s} ds = \sum_{\{w \in B(0,r) : h(w) = 0\}} \ln \left| \frac{r}{w} \right| = \frac{1}{2\pi} \int_0^{2\pi} \ln |h(re^{i\theta})| d\theta$$

- In particular, if $\sup_{|w|=1} |\ln h(w)| \leq M$, then

$$\sum_{\{z \in B(0,r) : h(w) = 0\}} (1 - |w|) \leq \sum_{\{z \in B(0,r) : h(w) = 0\}} \ln \left| \frac{1}{w} \right| \leq M.$$

Conformal map

- $\psi : \mathbb{D} \to \rho(H_0)$ conformal map s.t. $\psi(0) \in \rho(H_0)$.
- $h : \mathbb{D} \to \mathbb{D}$,

$$h(w) := \frac{\det_{\lceil\alpha\rceil}(I + V^{1/2} R_0(\psi(w))|V|^{1/2})}{\det_{\lceil\alpha\rceil}(I + V^{1/2} R_0(\psi(0))|V|^{1/2})}.$$

- ψ^{-1} extends diffeomorphically to $\mathbb{C} \setminus \Lambda_{\mathrm{crit}}(H_0)$.
- Koebe distortion theorem: $z = \psi(w)$

$$\Longrightarrow (1 - |w|) \approx \left|\frac{dw}{dz}\right| dist(z, \sigma(H_0)).$$

- $\exists \mu_j \geq 0$:

$$|\ln h(w)| \leq C(V) \prod_{z_j \in \Lambda_{\mathrm{crit}}(H_0) \cup \{\infty\}} |w - \psi^{-1}(z_j)|^{-\mu_j}.$$

Uniform resolvent bounds in Schatten spaces

Theorem

Let $H_0 \in \{(m^2-\Delta)^{s/2}, \alpha\cdot D + m\beta\}$. *There exists* $N : \rho(H_0) \to \mathbb{R}_+$ *with continuous extension to* $\mathbb{C}\setminus\Lambda_{\rm crit}(H_0)$ *s.t.*

a) *If* $s \geq 2d/(d+1)$ *and* $V \in L^{p\in[d/s,(d+1)/2]}$, *then*

$$\| V^{1/2} R_0(z) |V|^{1/2} \|_{\mathfrak{S}^{p(d-1)/(d-p)}} \leq N(z) \|V\|_{L^p}$$

b) *If* $s < 2d/(d+1)$ *and* $V \in L^{d/s} \cap L^{(d+1)/2}$, *then* $\forall \epsilon > 0$

$$\| V^{1/2} R_0(z) |V|^{1/2} \|_{\mathfrak{S}^{\max\{d+1, d/s+\epsilon\}}} \leq N(z) \|V\|_{L^d \cap L^{(d+1)/2}},$$

- The case $s = 2$ is due to Frank and Sabin [8].
- Proof uses Stein's interpolation theorem for analytic families of operators.
- Theorem is valid for more general operators.

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