Global stability of the normal state of superconductors in the presence of a strong electric current (after Almog-Helffer, R. Henry, Grebenkov-Helffer-Henry), Almog-Helffer-Pan) Part C.

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The starting point on the mathematical side was a paper of Yaniv Almog at Siam J. Math. Appl. . This work was continued in collaboration with Y. Almog and X. Pan by the analysis of specific toy models. Then, in collaboration with Y. Almog, we treat a rather general situation and show how the toy models are involved in the question. A new paper with Y. Almog and X. Pang treats the question of the localization of the solution in presence of a strong electric current.
The techniques used in this work appear to be useful in other contexts (Control theory (Beauchard-Henry-Helffer-Robbiano, Henry, Almog-Henry) or in magnetic resonance (Grebenkov, Grebenkov-Helffer-Henry)

Time-independent Ginzburg-Landau functional

We recall that the Ginzburg-Landau functional is given by

$$\mathcal{G}_{\kappa,H}[\psi,\mathbf{A}] = \int_\Omega \Big\{ |\nabla_{\kappa H \mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl} \mathbf{A} - \beta|^2 \Big\}\, dx\,,$$

with

- Ω simply connected,
- $(\psi, \mathbf{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$,
- $\nabla_{\mathbf{A}} = (\nabla + i\mathbf{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{div} \mathbf{A} = 0 \quad \text{in } \Omega\,,\ \ \mathbf{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Time Dependent Ginzburg-Landau equation.

Consider a superconductor placed at a temperature lower than the critical one. It is well-understood from numerous experimental observations, that a sufficiently strong current, applied through the sample, will force the superconductor to arrive at the normal state. To explain this phenomenon mathematically, we use the time-dependent Ginzburg-Landau model which is defined by the following system of equations, and will be referred to as (TDGL1) (Time Dependent Ginzburg-Landau equation).

If instead of applying an external magnetic field, we pass an electric current through Ω, we can no longer find (ψ, A) by looking for critical points of $\mathcal{G}$. Instead, we need to consider the time-dependent Ginzburg-Landau equations, originally obtained by Gor'kov and Eliashberg [28]. These are given, in terms of the energy $\mathcal{G}$, by

$$\frac{\partial \psi}{\partial t} + i\kappa\phi = \frac{\delta \mathcal{G}}{\delta \bar{\psi}} \tag{1a}$$

$$\frac{\sigma}{\kappa^2}\Big(\frac{\partial A}{\partial t} + \nabla\phi\Big) = \frac{\delta \mathcal{G}}{\delta A}\,. \tag{1b}$$

In the above, ϕ is the electric potential and σ is the dimensionless normal conductivity. If $\phi \equiv 0$, then, as can be easily verified, $\mathcal{G}$ becomes a Lyapunov function, and the solution will converge as $t \to \infty$ to a critical point of $\mathcal{G}$ [22]. If, however, an electric current is applied on the boundary, $\phi \not\equiv 0$, and the analysis of the problem becomes much more difficult.

(TDGL1)

$$\frac{\partial\psi}{\partial t} + i\phi\psi = (\nabla - iA)^2\psi + \psi\left(1 - |\psi|^2\right), \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{2a}$$

$$\kappa^2 \operatorname{curl}^2 A + \sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right) = \operatorname{Im}(\bar{\psi} \cdot (\nabla - iA)\psi), \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{2b}$$

$$\psi = 0, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c, \tag{2c}$$

$$(\nabla - iA)\psi \cdot \nu = 0, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i, \tag{2d}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = J, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c, \tag{2e}$$

$$\sigma\left(\frac{\partial A}{\partial t} + \nabla\phi\right)\cdot\nu = 0, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i. \tag{2f}$$

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \text{curl } A(t,x)\, ds = h_{ex}\,, \text{ on } \mathbb{R}_+\,, \quad (1g)$$

$$\psi(0,x) = \psi_0(x)\,, \qquad \text{in } \Omega\,, \quad (1h)$$

$$A(0,x) = A_0(x)\,, \qquad \text{in } \Omega\,, \quad (1i)\,.$$

In the above ψ denotes the order parameter, A is the magnetic potential, ϕ is the electric potential, κ denotes the Ginzburg-Landau parameter, which is a material property, and the normal conductivity of the sample is denoted by σ. ds denotes the induced measure on $\partial\Omega$. The domain $\Omega \subset\subset \mathbb{R}^2$, occupied by the superconducting sample, has in a subset of $\partial\Omega$, a smooth interface, denoted by $\partial\Omega_c$, with a conducting metal which is at the normal state.

We require that ψ would vanish on $\partial\Omega_c$ in (2c), and allow for a smooth current

$$J = hJ_r \,,$$

satisfying

$$(J1) \quad J_r \in C^2(\overline{\partial\Omega_c}), \tag{4}$$

to enter the sample in (2e).
We further require that

$$(J2) \quad \int_{\partial\Omega_c} J_r \, ds = 0 \,, \tag{5}$$

and

$$(J3) \quad \text{the sign of } J_r \text{ is constant on each connected component of } \partial\Omega_c \,. \tag{6}$$

We allow for $J_r \neq 0$ at the corners. (By convention, $J_r = 0$ on $\partial\Omega \setminus \partial\Omega_c$).

The rest of the boundary, denoted by $\partial\Omega_i$ is adjacent to an insulator. To simplify some of our arguments (or simply have a proof) we introduce the following geometrical assumption on $\partial\Omega$:

$$(R1) \left\{ \begin{array}{l} (a)\ \partial\Omega_i \text{ and } \partial\Omega_c \text{ are of class } C^3\,; \\ (b)\quad \text{Near each edge, } \partial\Omega_i \text{ and } \partial\Omega_c \text{ are flat} \\ \quad\ \text{and meet with an angle of } \frac{\pi}{2}\,. \end{array} \right. \tag{7}$$

We also require:

$$(R2) \qquad \text{Both } \partial\Omega_c \text{ and } \partial\Omega_i \text{ have two components.} \tag{8}$$

Figure 1 presents a typical sample with properties (R1) and (R2). Most wires would fall into the above class of domains.

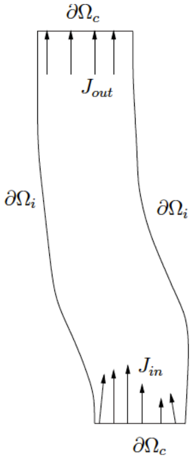

FIGURE 1. Typical superconducting sample. The arrows denote the direction of the current flow (J_{in} for the inlet, and J_{out} for the outlet).

We assume, for the initial conditions (2h,i), that

$$\psi_0 \in H^1(\Omega, \mathbb{C}) \text{ and } A_0 \in H^1(\Omega, \mathbb{R}^2)\,, \tag{9}$$

and:

$$\|\psi_0\|_\infty \leq 1\,. \tag{10}$$

We mainly consider Coulomb gauge solutions of (2):

$$\text{div } A = 0 \text{ in } \Omega,\ A \cdot \nu = 0 \text{ on } \partial\Omega\,. \tag{11}$$

Note that for the proof of existence of solutions it is better to consider first solutions in the Lorentz gauge:

$$\phi = \omega \text{ div } A\,.$$

Equivalent boundary conditions.

Instead of considering the boundary conditions (2e,f,g), it is possible to use an equivalent boundary condition where we prescribe instead the magnetic field at the boundary. By (2b,e,f), on each point on $\partial\Omega$, except for the corners, we have

$$\frac{\partial}{\partial\tau}\,\mathrm{curl}\,A(t,\cdot) = \frac{1}{\kappa^2}J(\cdot)\,, \tag{12}$$

where $\partial/\partial\tau$ denotes the tangential derivative along $\partial\Omega$ in the positive direction. For convenience we set

$$J_r(x) \equiv 0 \text{ on } \partial\Omega_i\,. \tag{13}$$

Thus, if we introduce on the boundary the function B_r by

$$\mathrm{curl}\,A(t,x) = h\,B_r(t,x) \quad \text{on } \partial\Omega\,, \tag{14}$$

where h denotes a parameter measuring the intensity of the magnetic field.

One can recover the magnetic field $B_r(t,\cdot)$

$$B_r(t,x) = h_r - \frac{1}{\kappa^2\,|\partial\Omega|}\int_{\partial\Omega} |\Gamma(\tilde{x},x)|\, J_r(\tilde{x}) ds(\tilde{x}) \;\text{ for } x \in \partial\Omega\,. \quad (15)$$

where $h_{ex} = hh_r$, $J = hJ_r$ and $|\Gamma(\tilde{x},x)|$ is the length inside the boundary between x and $\tilde{x}$.
This shows that $B_r(t,x) = B_r(x)$ on the boundary, hence independent of t.
Note also that

The magnetic field B_r is constant along each component of $\partial\Omega_i$. (16)

Hence the system (TGDL1) is equivalent to the system (TGDL2). We have the same equations except that (1e-1g) is replaced by

$$\text{curl } A(t,x) = hB_r(x)\,, \text{ on } \mathbb{R}_+ \times \partial\Omega\,, \tag{17}$$

where B_r is given by (15).
Of course functional spaces should be introduced to give a precise mathematical sense to this statement of equivalence.

(TDGL2)

$$\frac{\partial \psi}{\partial t} + i\phi\psi = (\nabla - iA)^2 \psi + \psi\left(1 - |\psi|^2\right), \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{18a}$$

$$\kappa^2 \operatorname{curl}{}^2 A + \sigma \left(\frac{\partial A}{\partial t} + \nabla \phi\right) = \operatorname{Im}(\bar{\psi} \cdot (\nabla - iA)\psi), \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{18b}$$

$$\psi = 0, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_c, \tag{18c}$$

$$(\nabla - iA)\psi \cdot \nu = 0, \qquad \text{on } \mathbb{R}_+ \times \partial\Omega_i, \tag{18d}$$

$$\operatorname{curl} A(t,x) = h\, B_r(x) \qquad \text{on } \partial\Omega \times \mathbb{R}_+, \tag{18e}$$

$$\psi(0,x) = \psi_0(x), \qquad \text{in } \Omega, \tag{18f}$$

$$A(0,x) = A_0(x), \qquad \text{in } \Omega. \tag{18g}$$

Conversely, a solution of (TGDL2) must satisfy (TGDL1) with

$$J_r = \kappa^2 \frac{\partial B_r}{\partial \tau} \text{on } \partial\Omega\,,$$

and

$$h_r = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} B_r(x) ds\,.$$

Stationary normal solutions.

If we assume time independence and a solution of (TDGL1) in the form $(0, A_n, \phi_n)$, we get for the magnetic and electric normal potentials A_n and ϕ_n. These equations are obtained by setting $\psi \equiv 0$ in (2b), yielding

$$\begin{cases} -c \;\text{ curl }^2 A_n + \nabla \phi_n = 0 & \text{ in } \Omega\,, \\ -\sigma \frac{\partial \phi_n}{\partial \nu} = J_r & \text{ on } \partial\Omega\,, \\ \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \text{ curl } A_n \, ds = h_r\,, & \end{cases}$$

in which $c = \kappa^2/\sigma$, J_r and h_r respectively denote some reference electric current and magnetic field.

If we fix the Coulomb gauge for A_n, we can prove the existence, uniqueness, and regularity of solutions to the above problem.
Note that ϕ_n is a solution of

$$\Delta\phi_n = 0$$

$$\int_\Omega \phi_n dx = 0\,,$$

and

$$-\sigma \frac{\partial \phi_n}{\partial \nu} = J_r\,.$$

This is Neumann but for a problem with corners ! H^2-regularity is OK when the angles are $\frac{\pi}{2}$.
See Kondratev, Grisvard, Dauge for these questions of regularity.

The next assumption (which can be expressed in term of J_r and h_r), is

$$(B) \quad B_n \neq 0 \text{ at the corners}\,, \tag{19}$$

where $B_n = \text{ curl } A_n$.

For some of the results, we assume for technical reasons

$$(C) \quad \nabla \phi_n \perp \partial\Omega \ \text{ on } B_n^{-1}(0) \cap \partial\Omega\,. \tag{20}$$

One possible mechanism which contributes to the breakdown of superconductivity by a strong current is the magnetic field induced by the current.
In the absence of electric current, we have seen (Giorgi-Phillips) that, when a sufficiently strong magnetic field is applied, the normal state, becomes the unique solution of the time-independent Ginzburg-Landau model.
For the time-dependent Ginzburg-Landau equations it was proved in Feireisl-Takac [22] that every solution must reach an equilibrium in the long-time limit. When combined with Giorgi-Phillips it follows that when the applied magnetic field is sufficiently large the normal state becomes globally stable.

No such result was available in the presence of electric currents.

The results in Feireisl-Takac [22] are based on the fact that, in the absence of currents, the Ginzburg-Landau energy functional serves as a Lyapunov functional.

The message of our analysis is that the magnetic field is not the only mechanism which forces the sample into the normal state when the electric current is sufficiently large.

Consider the reduced model where one neglects the induced magnetic field and set $A \equiv 0$ in (2). It has been proved in [38, 50, 3] that the normal state is at least locally stable when the current is sufficiently strong.
This corresponds to the model $-\Delta_{x,y} + icy$.
In Part B, we have computed a critical current where the normal state looses its local stability tends to the critical value for the reduced model [38] in the small conductivity limit $c \to 0$, or when $c \to \infty$. This result suggests that stability is being forced not only by the magnetic field that the current induces, but also by the potential term in (2a).

We now describe some of the results of a recent paper with Y. Almog, where we prove global stability of the normal state, as a solution of (2), for sufficiently large currents and mention also a recent paper by Almog-Helffer-Pan.
We have first to prove the global existence and uniqueness of solutions for (2) and obtain their regularity.
While these questions have previously addressed (cf. Chen-Hoffmann-Liang [12], [23], and [16] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention.

A non self-adjoint operator.

If $(0, \phi_n, \mathbf{A_n})$ is the normal steady state, let

$$\mathcal{L}_h = -\nabla^2_{hA_n} + i\, h\phi_n \,,$$

be defined over the domain

$$D(\mathcal{L}_h) = \{ u \in H^2(\Omega) \mid u|_{\partial\Omega_c} = 0 \,;\ \nabla u \cdot \nu|_{\partial\Omega_i} = 0 \} \,.$$

We prove that a proper bound on the resolvent of $\mathcal{L}_h$, which is the elliptic operator in (2a) linearized near $(0, hA_n, h\phi_n)$ gives the stability.

We want to prove global stability of the normal state, as a solution of (2), for sufficiently large currents. We begin by proving global existence and uniqueness of solutions for (2) and obtain their regularity. While these questions have previously addressed (cf. [12], [23], and [16] to name just a few references) the fact that the boundary is not smooth at the corners requires some additional attention.

We prove that if the current is strong enough, the magnetic field induced by this current forces the semigroup associated with (2) to become asymptotically a contraction. Let

$$\mu(h) = \inf_{\substack{u \in H^1(\Omega,\mathbb{C}) \\ u|_{\partial\Omega_c}=0\,;\,\|u\|_2=1}} \|\nabla_{hA_n} u\|_2^2\,.$$

This is simply the ground state energy of the magnetic Laplacian (selfadjoint part of $\mathcal{L}_h$). Under our assumptions, this is a Montgomery semi-classical Operator with Dirichlet-Neumann boundary condition.

Analysis of the linearized problem

Consider first the linearized version of (2a):

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u = 0\,, & \text{in } \mathbb{R}_+ \times \Omega\,, \\ (i\nabla + hA_n)u \cdot \nu = 0\,, & \text{on } \mathbb{R}_+ \times \partial\Omega_i\,, \\ u = 0\,, & \text{on } \mathbb{R}_+ \times \partial\Omega_c\,, \\ u(0,\cdot) = u_0(\cdot)\,, & \text{in } \Omega\,. \end{cases} \tag{21}$$

In the above

$$-\mathcal{L} = (\nabla - ihA_n)^2 + ih\phi_n + 1\,.$$

It is easy to show using integration by parts that for any $v \in D(\mathcal{L})$ we have

$$\langle v, \mathcal{L}v \rangle = \|\nabla_{hA_n} v\|_2^2 - \|v\|_2^2\,.$$

Hence

$$\langle v, \mathcal{L}v \rangle \geq (\mu - 1)\|v\|_2^2\,.$$

Note that if v is a ground state of $\mathcal{L}$ the above inequality becomes an identity. Hence, it follows that the operator $\mathcal{L}$ is accretive if and only if $\mu \geq 1$. Consequently, it is easy to show from the Lumer-Phillips Theorem (Theorem 8.3.5 in [13]) that the semigroup associated with (21) is a contraction semigroup if and only if $\mu \geq 1$. If $\mu > 1$ one can easily show that any solution of (21) decays exponentially fast (with a decay like $\exp -(\mu - 1)t$) and hence, that $u \equiv 0$ is asymptotically stable.

If we now consider the linearized part of (2b), (after taking its curl), we get the equation for the first variation w of curl A

$$\begin{cases} \sigma \partial_t w - \kappa^2 \Delta w = 0 \ \text{ in } \mathbb{R}^+ \times \Omega \,, \\ w(t,\cdot) = 0 \ \text{ on } \mathbb{R}^+ \times \partial\Omega \,, \\ w(0,\cdot) = w_0(\cdot) \ \text{ on } \Omega \,. \end{cases}$$

From the above we can conclude an $\mathcal{O}(e^{-\lambda_D ct})$-decay for $w(t,\cdot)$, where λ^D is the Dirichlet Laplacian and $c = \frac{\kappa^2}{\sigma}$.

The first result (Almog-Helffer) is the following

Theorem 1

Let (ψ, A, ϕ) denote a solution of (2) and (11) satisfying (10). Then, there exists $\gamma > 0$ for which whenever

$$\mu(h) > 1 + \frac{\gamma}{\kappa^2} + \frac{\gamma^2}{\kappa^4}\,, \tag{22}$$

there exist $C = C(\Omega, \kappa, c, \|\psi_0\|_2, \|A_0\|_2, h) > 0$ and $\lambda_m = \lambda_m(c, \kappa, \mu(h), \Omega) > 0$ such that, for all $t > 0$, we have:

$$\|\psi\|_2 + \|A - hA_n\|_2 + \|\phi - h\phi_n\|_2 \leq Ce^{-\lambda_m t}\,. \tag{23}$$

Note that (23) means that the semigroup associated with the linearized version of (2) is a contraction. Precise values of γ, λ_m, can be established in the large κ limit.

Theorem 2

Let $\nu \geq 0$. There exists $\kappa_0 > 0$ and $C_1 > 0$ such that, if for some $\kappa > \kappa_0$ we have

$$\sup_{\gamma \in \mathbb{R}} \|(\mathcal{L}_h - i\gamma - \nu)^{-1}\| < 1 - \frac{C_1}{\kappa^2}\,, \tag{24}$$

then, any solution of (2) must satisfy

$$\int_0^{\infty} e^{2\nu t}\, \|\psi(t,\cdot)\|_2^2 \, dt < \infty\,. \tag{25}$$

The above stability is proved in the large κ limit and we treat the same system, scaled with respect to the penetration depth, which is obtained by applying the transformation $x \to x/\kappa$ in (2).
As the resolvent of $\mathcal{L}_h$ in an arbitrary domain is difficult to control, we provide an estimate of its norm for large values of h, which can be applied for either large domains, or large κ values.

Large domains Ω_R

Our aim is to show that the norm of the resolvent can be controlled from two approximated problems, with constant current defined either in $\mathbb{R}^2$ or in $\mathbb{R}^2_+$ with Dirichlet boundary conditions. From resolvent estimates, together with the results of Almog-Helffer-Pan in [6, 4, 5] we deduce that the critical current, for which the normal state looses its local stability, can be approximated by the same critical current obtained for the above $\mathbb{R}^2_+$ problem. Before to state the result let us describe the toy models.

Two toy models

We now give the definitions of these model operators in $\mathbb{R}^2$ and $\mathbb{R}^2_+ = \{y > 0\}$. They were analyzed in Part B.
These models depend on two real parameters $c \neq 0$ and j.
The first one is

$$\mathcal{A}(\mathrm{j}, c) = D_x^2 + (D_y - \mathrm{j}x^2)^2 + ic\mathrm{j}y\,, \tag{26}$$

defined on

$$D(\mathcal{A}) = \{u \in L^2(\mathbb{R}^2) \,|\, \mathcal{A}u \in L^2(\mathbb{R}^2)\}\,. \tag{27}$$

It has empty spectrum and we have a good control of the resolvent depending only of the real part of the spectral parameter.

The second one is $\mathcal{A}_+(\mathrm{j}, c)$, which is defined (via the Lax-Milgram theorem) by the same differential formula of $\mathcal{A}$ but on the domain

$$D(\mathcal{A}_+) = \{u \in \tilde{V} \,:\, \mathcal{A}_+ u \in L^2(\mathbb{R}^2_+, \mathbb{C})\}, \tag{28}$$

where

$$\tilde{V} = H_0^{1,\mathrm{mag}}(\mathbb{R}^2_+, \mathbb{C}) \cap L^2(\mathbb{R}^2_+, \mathbb{C}; y\, dxdy)\,. \tag{29}$$

Here the analysis of the spectrum is more difficult. The guess is that it is non-empty. This is only proven for $|c|$ large enough or small enough.

Towards the last theorem

Set, for $z \in \bar{\Omega}$,

$$\mathfrak{j}(z) := h|\nabla B_n(z)| = \frac{h}{c}|\nabla \phi_n(z)|\,, \tag{30}$$

and then define,

$$\mathcal{A}(z) = \mathcal{A}(\mathfrak{j}(z), c) \quad ; \quad \mathcal{A}_+(z) = \mathcal{A}_+(\mathfrak{j}(z), c) \tag{31}$$

Under the above assumptions $B_n^{-1}(0)$ is either empty, or else consists of a single curve Γ connecting between the two connected components of $\partial\Omega_c$.

We treat the second case. We denote the two points of intersection by z_1 and z_2 and then set

$$\nu_m(z_1, z_2, c) = \min_{i=1,2} \inf_{\lambda \in \sigma(\mathcal{A}_+(z_i))} \mathrm{Re}\lambda \,. \tag{32}$$

Large domain limit

Let then $R > 0$. We denote by Ω_R the image of Ω under the dilation $x \to R\,x$ We assume that the domain Ω has the property (R1)-(R2) and that assumptions (J1)-(J3), (B) and (C) are met. Denote the transformed electric field by ϕ_R. It satisfies the problem

$$\begin{cases} \Delta\phi_R = 0 & \text{in } \Omega_R\,, \\ \frac{\partial \phi_R}{\partial \nu} = -\frac{J_R(x)}{\sigma} & \text{on } \partial\Omega_R\,, \end{cases}$$

where

$$J_R(x) = J_r(x/R)\,.$$

Note that

$$\phi_R(x) = R\,\phi_n(x/R)\,.$$

The transformed magnetic potential, which we denote by A_R then satisfies

$$A_R(x) = R^2\,A_n(x/R)\,.$$

Let then

$$\mathcal{L}_h^R = -\nabla^2_{hA_R} + ih\phi_R\,, \tag{33}$$

and let

$$\mu(R) = \inf_{\lambda\in\sigma(\mathcal{L}_h^R)} \mathbb{R}e\lambda \ \text{ and } \quad \mu_\infty = \liminf_{R\to\infty} \mu(R)\,. \tag{34}$$

We can now state

Theorem 3

Under the previous assumptions,

$$\mu_\infty = \nu_m\,.$$

Furthermore, let $\nu < \mu_\infty$. Then, $\exists\ R_0,\ C,$ such that, for $R \geq R_0$,

$$\sup_{\gamma\in\mathbb{R}} \|(\mathcal{L}_h^R - \nu - i\gamma)^{-1}\| \leq$$

$$\max\Big(\sup_{z_0\in\Gamma}\|(\mathcal{A}(z_0)-\nu)^{-1}\|,\sup_{\substack{\gamma\in\mathbb{R}\\ i=1,2}}\|(\mathcal{A}_+(z_i)-\nu-i\gamma)^{-1}\|\Big)\Big(1+\frac{C}{R^{1/4}}\Big) + \frac{C}{R^{1/4}}\,. \quad (35)$$

One can deduce from (35) an upper bound for the critical current where the normal state $(0, hA_n, h\phi_n)$ becomes globally stable. Let

$$\mathrm{j}_m = \inf_{z\in\Gamma} \mathrm{j}(z)\,, \tag{36a}$$

and

$$\mathrm{j}_+ = \inf_{i=1,2} \mathrm{j}(z_i)\,. \tag{36b}$$

When the domain size is multiplied by R, the resolvent norm of $\mathcal{L}_h$ is given by the left-hand-side of (35). By (24) it then follows that if R and κ are sufficiently large, and if

$$\mathrm{j}_m > \|\mathcal{A}^{-1}(1,c)\|^{3/2} \tag{37a}$$

and

$$\mathrm{j}_+ > \sup_{\gamma\in\mathbb{R}} \|(\mathcal{A}_+(1,c) - i\gamma)^{-1}\|^{3/2}\,, \tag{37b}$$

then the normal state must be globally stable. The above conditions serve as an upper bound for the critical current where the normal state becomes globally stable.

On the semiclassical side

This corresponds to the spectral analysis of

$$\sum_j (\hbar D_{x_j} - A_j)^2 + i\hbar\phi(x)\,,$$

in the limit $\hbar \to 0$. With $\phi = 0$, this analysis plays an important role in the analysis of the superconductivity. In the above questions, we have $\nabla\phi \cdot \nabla \text{ curl } A = 0$ and the zeros of $\text{curl } A$ consists in a curve Γ joining two points of the boundary where the Dirichlet condition is assumed.

When $\mathbf{A} = \mathbf{0}$, we look more simply at

$$\mathcal{A}_h := -h^2\Delta + i\phi\,.$$

This is what we discuss now.

More recent results by R. Henry (2013), see also Almog-Henry (2015)

We first focus on the case where the potential ϕ has no critical point:

Theorem 4

Let $n \geq 1$ and $\phi \in \mathcal{C}^\infty(\bar{\Omega}; \mathbb{R})$ with $\nabla\phi(x) \neq 0$. Let

$$\partial\Omega_\perp = \{x \in \partial\Omega : \nabla\phi(x) \times \vec{n}(x) = 0\}\,. \tag{38}$$

Assume that $\partial\Omega_\perp \neq \emptyset$. Let $\mu_1 < 0$ be the rightmost zero of the Airy function , and let

$$J_m = \min_{x \in \partial\Omega_\perp} |\nabla\phi(x)|\,. \tag{39}$$

Then we have

$$\underline{\lim}_{h \to 0} \frac{1}{h^{2/3}} \inf \mathrm{Re}\, \sigma(\mathcal{A}_h) \geq \frac{|\mu_1|}{2} J_m^{2/3}\,. \tag{40}$$

The last inequality can be shown to be an equality under some additional assumption.

Theorem 4 continued

Moreover, for every $\varepsilon > 0$, $\exists h_\varepsilon \in (0, h_0)$ and $\exists C_\varepsilon > 0$ such that

$$\forall h \in (0, h_\varepsilon), \quad \sup_{\substack{\gamma \leq |\mu_1| J^{2/3}/2, \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_h - (\gamma - \varepsilon) h^{2/3} - i\nu)^{-1}\| \leq \frac{C_\varepsilon}{h^{2/3}} . \tag{41}$$

Assume now that $\partial\Omega_\perp = \emptyset$, then

$$\lim_{h \to 0} \frac{1}{h^{2/3}} \inf \mathrm{Re}\, \sigma(\mathcal{A}_h) = +\infty ,$$

and for all $\omega \in \mathbb{R}$, $\exists h_\omega > 0$ and $C'_\omega > 0$ such that

$$\forall h \in (0, h_\omega), \quad \sup_{\substack{\gamma \leq \omega, \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_h - \gamma h^{2/3} - i\nu)^{-1}\| \leq \frac{C'_\omega}{h^{2/3}} . \tag{42}$$

This result is essentially a reformulation of those stated by Y. Almog, but the proof of Henry is based on locally approximating models. That $\frac{|\mu_1|}{2} J_m^{2/3}$ is the exact limit for $h^{-2/3} \inf \mathrm{Re}\, \sigma(\mathcal{A}_h)$ as $h \to 0$ was till recently open (except in the $(1D)$-case). This has been solved in a new paper (2015) by Almog-Henry (see also Grebenkov-Helffer (2016)).
In the case where the potential V has critical points in the interior of Ω, the spectrum of $\mathcal{A}_h$ is expected to behave differently. The following statement shows that the quantity $\inf \mathrm{Re}\, \sigma(\mathcal{A}_h)$ is no longer determined by the behavior at the boundary, but by the shape of the potential near the critical points.

Theorem 5

Let ϕ be a Morse function on $\bar{\Omega}$ and $x_1^c, \dots, x_p^c$ the critical points of ϕ. Let

$$\kappa_k = \sum_{j=1}^{n} \sqrt{|\lambda_j^k|}, \text{ where } \{\lambda_j^k\}_{j=1,\dots,n} = \sigma(\mathrm{Hess}\phi(x_k^c))\,. \tag{43}$$

Let $\kappa = \min_{k=1,\dots,p} \kappa_k$, and assume that, if $\kappa_k = \kappa$, then $\forall \ell \neq k$,

$$\phi(x_k^c) \neq \phi(x_\ell^c)\,. \tag{44}$$

Then,

$$\lim_{t \to 0} \frac{1}{h} \inf \mathrm{Re}\, \sigma(\mathcal{A}_h) = \frac{\kappa}{2}\,. \tag{45}$$

Moreover, $\forall \varepsilon > 0$, $\exists h_\varepsilon \in (0, h_0)$ and $\exists C_\varepsilon > 0$ s.t.

$$\forall h \in (0, h_\varepsilon), \qquad \sup_{\substack{\gamma \leq \kappa/2, \\ \nu \in \mathbb{R}}} \|(\mathcal{A}_h - (\gamma - \varepsilon)h - i\nu)^{-1}\| \leq \frac{C_\varepsilon}{h}\,. \tag{46}$$

The assumption (44) is meant to avoid any resonance phenomenon between two wells. Note that, unlike in Theorem 4, we give here the exact limit for $h^{-1}\inf \mathrm{Re}\,\sigma(\mathcal{A}_h)$.

The previous theorems enable us to state some decay estimates for the semigroup associated with $\mathcal{A}_h$.

Corollary

For all $\varepsilon > 0$, there exists $h_\varepsilon \in (0, h_0)$ and $M_\varepsilon > 0$ such that:

(i) Under Assumptions of Theorem 4, $\forall h \in (0, h_\varepsilon)\,,\ \forall t > 0$

$$\|e^{-t\mathcal{A}_h}\|_{\mathcal{L}(L^2(\Omega))} \leq M_\varepsilon \exp(-(|\mu_1| J_m^{2/3}/2 - \varepsilon) h^{2/3} t)\,. \qquad (47)$$

(ii) Under Assumptions of Theorem 5, $\forall h \in (0, h_\varepsilon)\,,\ \forall t > 0$

$$\|e^{-t\mathcal{A}_h}\|_{\mathcal{L}(L^2(\Omega))} \leq M_\varepsilon \exp(-(\kappa/2 - \varepsilon) h t)\,. \qquad (48)$$

This corollary follows by using a refined, quantitative version of the Gearhardt-Prüss Theorem due to Helffer-Sjöstrand.

Decay estimates (after Almog-Helffer-Pan (2015)

We now focus attention on the exponential decay of ψ in regions where $|B_n| > 1$. For steady-state solutions of (2) in the absence of electric current ($J = 0$) we may set $\phi \equiv 0$ and the magnetic field is then constant on the boundary. The exponential decay of ψ away from the boundary has been termed "surface superconductivity" and has extensively been studied (see Fournais-Helffer (book)). More recently, the case of a non-constant magnetic field has been studied as well (Attar (2013-2014), Helffer-Kachmar (2015),...). In these works ϕ still identically vanishes but nevertheless $\nabla B_n \neq 0$ in view of the presence of a current source term $\text{curl } h_{ex}$ in (2b). In particular in Helffer-Kachmar (2015, 2016) it has been established, in the large κ limit for the case $1 \ll h \ll \kappa$ that ψ is exponentially small away from $\Gamma = B_n^{-1}(0)$. We refer to Part A (with $\beta = B_n$).

As mentioned above, in the absence of electric current the time-dependent solutions is of lesser interest, since it converges to a steady-state solution [41, 22]. This result has been obtained in [22] by using the fact that the Ginzburg-Landau energy functional is a Lyapunov function in this case. In contrast, when $J \neq 0$ this property of the energy functional is lost, and convergence to a steady-state is no-longer guaranteed. In [**?**] the global stability of the normal state has been established for $h = \mathcal{O}(\kappa)$ in the large κ limit. Solutions of (2), for much weaker current densities ($J \sim \mathcal{O}(\ln \kappa)$) have been addressed in [54]. Whereas simplified model for strong magnetic fields have been addressed in [19, 33].

For the same limit, we explore now the behavior of the solution for $1 < h \ll \kappa$, and establish exponential decay of ψ in every subdomain of Ω where $|B_n| > 1$. We do that for both steady-state solutions (whose existence we need to assume) and time-dependent ones. We also study the large-domain limit, where we obtain weaker results for steady-state solutions only. Let, for $j = 1, 2$,

$$\omega_j = \{x \in \Omega \,:\, (-1)^j B_n(x) > 1\}\,, \tag{49}$$

where B_n is the normal magnetic field.
Our first decay theorem concerns steady state solutions and their exponential decay, in certain subdomains of Ω, in the large κ limit.

Steady State Main Theorem

Let for $\kappa \geq 1$, $(\psi_\kappa, A_\kappa, \phi_\kappa)$ be a time-independent solution of (2). Suppose that for some $j \in \{1, 2\}$ we have

$$1 < |h_j| \,. \tag{50}$$

Then, for any compact set $K \subset \omega_j \cup \partial\Omega_c$, there exist $C > 0$, $\alpha > 0$, and $\kappa_0 \geq 1$, such that for any $\kappa \geq \kappa_0$ we have

$$\int_K |\psi_\kappa(x)|^2 \, dx \leq C e^{-\alpha\kappa} \,. \tag{51}$$

If, in addition,

$$\frac{1}{\Theta_0} < |h_j| \,, \tag{52}$$

then (51) is satisfied for any compact subset $K \subset \omega_j \cup (\overline{\omega_j} \cap \partial\Omega_c)$.

In addition, we establish a weaker decay of ψ_κ in all Ω.

Proposition

Under the assumptions of the theorem, there exists $C(J,\Omega) > 0$ such that, for $\kappa \geq 1$,

$$\|\psi_\kappa\|_{L^2(\Omega,\mathbb{C})} \leq C(J,\Omega)\,(1 + c^{-1/2})^{1/3}\kappa^{-1/6}\,. \qquad (53)$$

Steady State Main Theorem is extended to the time dependent case in the following way.

Theorem

Let $(\psi_\kappa, A_\kappa, \phi_\kappa)$ denote a time-dependent solution of (2). Assuming $c = 1$, under the conditions of the main theorem and in addition on the initial condition, for any compact set $K \subset \omega_j \cup \partial\Omega_c$ there exist $C > 0$, $\alpha > 0$, and $\kappa_0 \geq 1$, such that for any $\kappa \geq \kappa_0$ we have

$$\limsup_{t\to\infty} \int_K |\psi_\kappa(t,x)|^2 \, dx \leq C e^{-\alpha\kappa} \,. \tag{54}$$

Finally, we consider steady-state solutions of (2) in the large domain limit. More precisely, we set $\kappa = c = 1$ and stretch Ω by a factor of $R \gg 1$. Let Ω^R be the image of Ω under the map $x \to Rx$. We consider again steady-state solutions of (2) in Ω^R:

$$\Delta_A \psi + \psi \left(1 - |\psi|^2\right) - i\phi\psi = 0 \quad \text{in } \Omega^R\,, \tag{55a}$$

$$\text{curl}\,^2 A + \nabla\phi = \mathrm{Im}\,(\bar{\psi}\,\nabla_A \psi) \quad \text{in } \Omega^R\,, \tag{55b}$$

$$\psi = 0 \quad \text{on } \partial\Omega_c^R\,, \tag{55c}$$

$$\nabla_A \psi \cdot \nu = 0 \quad \text{on } \partial\Omega_i^R\,, \tag{55d}$$

$$\frac{\partial\phi}{\partial\nu} = \frac{F(R)}{R} J \quad \text{on } \partial\Omega_c^R\,, \tag{55e}$$

$$\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \partial\Omega_i^R\,, \tag{55f}$$

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega^R} \text{curl}\, A(x)\, ds = F(R) h_{ex}\,. \tag{55g}$$

We study the above problem in the limit $R \to \infty$, supposing $F(R) \gg 1$ in that limit, where we establish the following result:

Proposition

Let (ψ, A, ϕ) denote a solution of (55) in Ω^R. Then, there exists a compact set with non empty interior $K \subset \Omega$, $C > 0$, $R_0 > 0$, and $\alpha > 0$, such that for any $R > R_0$ we have

$$\int_{K_R} |\psi(x)|^2 \, dx \leq Ce^{-\alpha R} \,, \tag{56}$$

where K_R is the image of K under the map $x \to Rx$.

Note that in the above proposition h_{ex} must be of the same order as J, otherwise $(0, A_n, \phi_n)$ would be the unique solution. Physically (56) demonstrates that there is a significant portion of the superconducting sample which remains, practically, at the normal state, for current densities which may be very small. This result stands in contrast with what one finds in standard physics handbooks [47] where the critical current density, for which the fully superconducting state looses its stability, is tabulated a material property. However, our results suggest that the critical current depends also on the geometry of the superconducting sample. In fact, according to the large domain proposition, this current density must decay in the large domain limit. In two-dimensions, our result suggests that one should search for a critical current.

We note that this Proposition for large domains is certainly not optimal. In fact, we expect the following conjecture to be true.

Conjecture

Under the conditions of Proposition 60, for any compact set $K \subset \Omega \setminus B_n^{-1}(0)$, there exist $R_0 > 0$, $C > 0$, and $\alpha > 0$, such that for any $R > R_0$ (56) is satisfied for any solution (ψ, A, ϕ) of (55) in Ω^R.

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