Ginzburg-Landau functional (time independent case) Part A. Ginzburg-Landau with variable exterior magnetic fields. (after Pan-Kwek, Attar, Helffer-Kachmar)

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Abstract

[Part A] We study the infimum of the Ginzburg-Landau functional in a two dimensional simply connected domain and with an external magnetic field allowed to vanish along a smooth curve. We obtain an energy asymptotics which is valid when the Ginzburg-Landau parameter is large and the strength of the external field is below the third critical field. We also discuss the localization of the minimizers. Compared with the known results when the external magnetic field does not vanish, we show in this regime a concentration of the energy near the zero set of the external magnetic field. The main results have been obtained by X.B. Pan–K.H. Kwek, Fournais-Helffer, K. Attar, B.Helffer-A.Kashmar.

Introduction

The Ginzburg-Landau functional is a model describing the response of a superconducting material to an applied magnetic field through the qualitative behavior of the minimizing/critical configurations. The mathematically rigorous analysis of such configurations led to a vast literature and to many mathematically challenging questions, with the aim to recover what physicists have already observed through experiments or heuristic computations. See the book by De Gennes for an introduction to the physics of superconductivity, and the two monographs (Fournais-Helffer, Sandier-Serfaty for the mathematical progress on this subject).

Much of the mathematical literature concerns samples in the form of a long cylinder or a thin film subject to a constant magnetic field. The direction of the magnetic field is parallel to the cylinder's axis (for cylindrical samples) or perpendicular to the plane of the thin film (for thin film samples). For such samples, we have the following behavior:

- For large values of the intensity of the magnetic field, the magnetic field penetrates the sample which is in a normal (non-superconducting) state.
- Decreasing the intensity of the magnetic field gradually past a critical value H_{c_3}, superconductivity nucleates along the boundary of the sample; the bulk of the sample remains in a normal state ; this is the phenomenon of surface superconductivity (see Saint-James—De Gennes).
- Decreasing the field further, superconductivity is restored in the bulk of the sample; the magnetic field may penetrate the sample along point defects called *vortices*; such vortices indicate regions of the sample that remain in the normal state (see Sandier-Serfaty).

The Ginzburg-Landau functional.

Let us describe the mathematical problem. It is naturally posed for domains in $\mathbb{R}^3$, but for cylindrical domains in $\mathbb{R}^3$, it is natural to consider a functional defined in a domain $\Omega \subset \mathbb{R}^2$, where Ω is the cross-section of the cylinder. This is why we consider models in $\mathbb{R}^2$. We assume Ω simply connected.

The Ginzburg-Landau functional is defined by

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(|\nabla_{\kappa H \mathbf{A}} \psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) d\mathbf{x} + (\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - \beta|^2 \, d\mathbf{x} \,. \quad (1)$$

Here the function ψ is called the order parameter (or sometimes the wave function) and $\mathbf{A}$ is a magnetic potential.

For $\mathbf{A} = (A_1, A_2)$, curl $\mathbf{A} = \partial_x A_2 - \partial_y A_1$ and $\nabla_{\kappa H \mathbf{A}}$ denotes the magnetic gradient: $\nabla + i\kappa H \mathbf{A}$. The symbol $\beta \in L^2$ is called the external or applied magnetic field.
The parameter $\kappa > 0$ (the Ginzburg-Landau parameter) depends on the material, and $H > 0$ measures the strength of the external magnetic field. We consider the asymptotic regime $\kappa \to +\infty$, which corresponds to strong type II samples.

We will sometime write $\mathcal{G} = \mathcal{G}_{\kappa,H}$, to mention the parameters involved. The natural domain of the functional is $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$. However, due to the gauge invariance of $\mathcal{G}$, we can restrict the functional to the smaller set $H^1(\Omega,\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, where

$$H^1_{\mathrm{div}}(\Omega) = \left\{ \mathbf{V} \in H^1(\Omega,\mathbb{R})^2 \;\middle|\; \text{ div } \mathbf{V} = 0 \text{ in } \Omega\,,\ \mathbf{V}\cdot\nu = 0 \text{ on } \partial\Omega \right\}. \tag{2}$$

We define the Ginzburg-Landau ground state energy to be the infimum of the functional, i.e.

$$\mathcal{E}_{min}(\kappa, H) := \inf_{(\psi,\mathbf{A})\in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)} \mathcal{G}_{\kappa,H}(\psi, \mathbf{A})\,. \tag{3}$$

If $\mathbf{F}$ is the unique magnetic potential in $H^1_{\mathrm{div}}(\Omega)$ such that:

$$\mathrm{curl}\ \mathbf{F} = \beta\,,$$

we observe that: $\mathcal{G}_{\kappa,H}(\psi \equiv 0, \mathbf{F}) = 0$, which implies the inequality:

$$\mathcal{E}_{min}(\kappa, H) \leq 0\,. \tag{4}$$

The pair $(0, \mathbf{F})$ is called *normal state* in Physics and we will, in particular, study when we have equality or strict inequality in (4).

Minimizers and Ginzburg-Landau equations

As Ω is bounded, the existence of a minimizer is rather standard. A minimizer should satisfy the Euler-Lagrange equation, which is called in this context the Ginzburg-Landau system and reads:

$$\left.\begin{aligned} -\Delta_{\kappa H \mathbf{A}}\psi &= \kappa^2(1-|\psi|^2)\psi\,, \\ \text{curl}\,\left(\text{curl}\,\mathbf{A}-\beta\right) &= -\frac{1}{\kappa H}\mathrm{Re}\left(i\overline{\psi}\,\nabla_{\kappa H\mathbf{A}}\psi\right), \end{aligned}\right\} \quad \text{in} \quad \Omega\,, \tag{5a}$$

$$\left.\begin{aligned} \nu\cdot\nabla_{\kappa H\mathbf{A}}\psi &= 0\,, \\ \text{curl}\,\mathbf{A}-\beta &= 0\,, \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{5b}$$

Here, $-\Delta_{\kappa H \mathbf{A}}$ is the magnetic Laplacian:

$$-\Delta_{\kappa H \mathbf{A}} := (D_x + \kappa H A_1)^2 + (D_y + \kappa H A_2)^2 \,, \text{ with } D_x = -i\partial_x \,,\; D_y = -i\partial_y \,,$$

and

$$\text{curl }^2 \mathbf{A} = (\partial_y(\text{ curl } \mathbf{A}), -\partial_x(\text{ curl } \mathbf{A}))\,.$$

The analysis of the system (5) can be performed by PDE techniques. We note that this system is nonlinear, that $H^1(\Omega)$ is, when Ω is bounded and regular in $\mathbb{R}^2$, compactly embedded in $L^p(\Omega)$ for all $p \in [1, +\infty)$.

Actually, the nonlinearity is weak in the sense that the principal part is a linear elliptic system. One can show in particular that the solution in $H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of the elliptic system (5) is actually, when Ω is regular, in $C^\infty(\overline{\Omega})$.

Basic properties for solutions of the Ginzburg-Landau equations

The first important property which is a consequence of the maximum principle is

Proposition

If $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1(\Omega, \mathbb{R}^2)$ is a (weak) solution to (5), then

$$\|\psi\|_{L^\infty(\Omega)} \leq 1\,. \tag{6}$$

Using this Proposition, we can get various a priori estimates on solutions to the Ginzburg-Landau equations (5), which play an important role in the whole theory.

Proposition: a priori estimates on the critical points

Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth, and let $\beta \in L^2(\Omega)$ be given. Then for all $p \geq 2$, there exists $C = C(p) > 0$ such that for all solutions $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\rm div}(\Omega)$ to (5), we have

$$\|\nabla^2_{\kappa H \mathbf{A}} \psi\|_p \leq \kappa^2 \|\psi\|_p \,, \tag{7}$$

$$\|\nabla_{\kappa H \mathbf{A}} \psi\|_2 \leq \kappa \|\psi\|_2 \,, \tag{8}$$

$$\| \operatorname{curl} \mathbf{A} - \beta\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa H} \|\psi\|_\infty \, \|\nabla_{\kappa H \mathbf{A}} \psi\|_p \,. \tag{9}$$

It is also useful to have Hölder estimates.

The Giorgi-Phillips Theorem for minimizers

We observe that $(0, \mathbf{F})$ is a trivial critical point of the functional $\mathcal{G}$, i.e., a trivial solution of the Ginzburg-Landau system (5). The pair $(0, \mathbf{F})$ is often called the *normal state* or *normal solution*. It is natural to discuss—as a function of H—whether this pair is a local or global minimizer. When H is large, one will show that this solution is effectively the unique global minimizer. One says that in this case the superconductivity is destroyed. In other words, the order parameter is identically zero in Ω. Let us give a rather simple proof of this result that roughly says that $(0, \mathbf{F})$ is the unique minimizer of the functional when the strength of the exterior magnetic field is sufficiently large. We actually show a stronger result for all the solutions of the associated Ginzburg-Landau system.

So we assume that we have a **nonnormal** stationary point $(\psi, \mathbf{A})$ for $\mathcal{G}$, that is a solution $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ of (5) satisfying

$$\int_\Omega |\psi(\mathbf{x})|^2 \, d\mathbf{x} > 0 \,. \qquad (10)$$

By (8), (9), and (6), and using a standard inequality on the curl $-$ div system for controlling $\|\mathbf{A} - \mathbf{F}\|^2$ in Ω by $\|$ curl $\mathbf{A} - \beta\|^2$, we get

$$\|\nabla_{\kappa H \mathbf{A}}\psi\|_2^2 + (\kappa H)^2 \|\mathbf{A} - \mathbf{F}\|_2^2 \leq C_\Omega \kappa^2 \|\psi\|_2^2 \,. \qquad (11)$$

Writing $\mathbf{A} = \mathbf{A} - \mathbf{F} + \mathbf{F}$ and implementing (6) and (11) give

$$\int_\Omega |(\nabla + i\kappa H\mathbf{F})\psi|^2 \, d\mathbf{x} \leq 2C_\Omega \kappa^2 \int_\Omega |\psi(\mathbf{x})|^2 \, d\mathbf{x} \,. \tag{12}$$

Since ψ satisfies (10), we obtain

$$\lambda_1^N(H\kappa\mathbf{F}) \leq 2C_\Omega \kappa^2 \,, \tag{13}$$

where $\lambda_1^N(H\kappa\mathbf{F})$ denotes the ground state energy of the Neumann realization of $-\Delta_{H\kappa\mathbf{F}}$ in Ω.

We observe that $\lambda_1^N(H\kappa\mathbf{F}) > 0$. So by combining an analysis in the small B regime (perturbation theory) and for large B, and the continuity of $B \mapsto \lambda_1^N(B\mathbf{F})$, we get the existence of $C_0 > 0$ s.t.

$$\lambda_1^N(H\kappa\mathbf{F}) \geq \frac{1}{C_0} \min(H\kappa, (H\kappa)^2) . \tag{14}$$

Thus, we find that if a nontrivial stationary point $(\psi, \mathbf{A})$ exists, then

$$H \leq C\,(1+\kappa) .$$

We have obtained:

Giorgi-Phillips Theorem

Let $\Omega \subset \mathbb{R}^2$ be smooth, bounded, and simply connected, and let β in (5) be continuous and satisfy

$$\beta(x) > 0\,, \qquad \forall x \in \bar{\Omega}\,.$$

Then there exists C such that if

$$H \geq C \max\{\kappa, 1\}\,,$$

then the pair $(0, \mathbf{F})$ is the unique solution to (5) in $H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$.

We have used in the proof:

Lu-Pan Theorem

$$\lambda_1^N(BF) = B \min(b, \Theta_0 b') + o(B)\,,$$

where $\Theta_0 \in (0,1)$, $b = \inf_{x\in\Omega} \beta(x)$ and $b' = \inf_{x\in\partial\Omega} \beta(x)$.

Two models are indeed involved in the proof by localization: the model with constant magnetic fields

$$(D_x - \frac{B}{2}\beta(x_j,y_j)y)^2 + (D_y + \frac{B}{2}\beta(x_j,y_j)x)^2\,,$$

in $\mathbb{R}^2$ and the Neumann realization of the same operator in $\mathbb{R}^2_+$. The bottom of the spectrum of the first one is $B|\beta(x_j,y_j)|$ and the bottom of the spectrum of the second one is $\Theta_0 B|\beta(x_j,y_j)|$.

Remark

In this form this theorem is due to Lu-Pan (1999). Many improvements concerning the control the remainder term $o(B)$ have been obtained. See the book of Fournais-Helffer (2010) and references therein and for more recent references Raymond, Raymond-Vu-Ngoc (2014), Helffer-Kordyukov (2014) and a recent book by N. Raymond (2016).

Remark

The Giorgi-Phillips statement is the starting point of the analysis of the third critical field corresponding to the transition between normal minimizers and non-normal minimizers. We refer to the books of Fournais-Helffer (2010) and Sandier-Serfaty (2007) for a detailed analysis of the behavior of these critical fields and references therein.

Vanishing exterior fields

We begin to discuss the case when β vanishes in Ω but

$$|\beta(\mathbf{x})| + |\nabla\beta(\mathbf{x})| > 0\,. \tag{15}$$

More precisely, introducing

$$\Gamma = \beta^{-1}(0)\,,$$

one has the theorem:

Pan-Kwek Theorem

$$\lim_{B \to +\infty} \frac{\lambda_1^N(B\mathbf{A})}{B^{\frac{2}{3}}} = [\alpha_1(\beta)]^{\frac{2}{3}} \,, \tag{16}$$

where

$$\alpha_1(\beta) = \min\left\{ \frac{1}{2}\lambda_0^{\frac{3}{2}} \inf_{\mathbf{x}\in\Omega\cap\Gamma} |\nabla\beta(\mathbf{x})| \,,\ \inf_{\mathbf{x}\in\partial\Omega\cap\Gamma} \zeta(\vartheta(\mathbf{x}))^{\frac{3}{2}} |\nabla\beta(\mathbf{x})| \right\} \,, \tag{17}$$

where λ_0 is the lowest eigenvalue of the Montgomery operator (to be defined later) and $\vartheta(\mathbf{x})$ denotes the angle between curl β and the tangent vector of $\partial\Omega$ at $\mathbf{x}$ and $\zeta(\vartheta)$ denotes the lowest eigenvalue of the Neumann realization of $-\Delta_{\mathbf{A}_\vartheta}$ in $\mathbb{R}^2_+$ with $\mathbf{A}_\vartheta = -\frac{|\mathbf{x}|^2}{2}(\cos\vartheta, \sin\vartheta)$.

As a consequence, we can extend the Giorgi-Phillips theorem to this case.

GP-Theorem with vanishing magnetic field

Let $\Omega \subset \mathbb{R}^2$ be smooth, bounded, and simply connected, and let β be in $C^\infty(\overline{\Omega})$ and satisfying (15). Then there exists C s.t. if

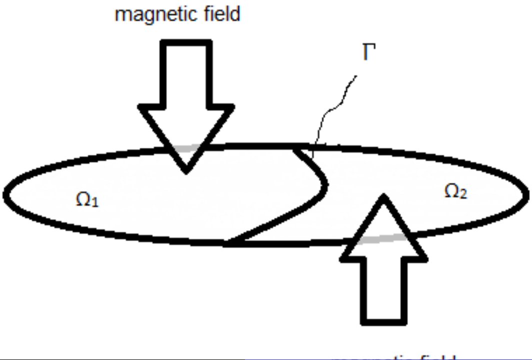
$$H \geq C \max\{\kappa^2, 1\}\,,$$

then the pair $(0, \mathbf{F})$ is the unique solution to (5) in $H^1(\Omega) \times H^1_{\rm div}(\Omega)$.

Notice that a more precise statement is given in Pan-Kwek in the limit κ large (see also the talk of J.P. Miqueu).

Minimal Energy asymptotics

Samples submitted to variable magnetic fields are considered in the physical literature. See for example the talk of J.P. Solovej. The reduction from the N-body problem to the Ginzburg-Landau functional does not lead to a constant applied magnetic field.
For the sake of illustrating the results, let us consider a thin film sample placed horizontally (see Figure 1).



The region occupied by the sample is decomposed into Ω_1 and Ω_2 separated by a smooth curve Γ. We suppose that β vanish along the smooth curve Γ. We have the following picture:

- For very large values of the intensity of the magnetic field, the sample is in a normal state (Giorgi-Phillips, Pan-Kwek).
- Decreasing the intensity of the magnetic field gradually past a critical value H_{c_3}, superconductivity nucleates along Γ; the rest of the sample remains in a normal state ; this is in contrast of the phenomenon of surface superconductivity observed for samples subject to a constant magnetic field.

When $H \gg \kappa^2$, then Giorgi-Phlillips like theorem says that the energy is the energy of the normal state, i.e. 0.
When decreasing H, various regimes appear depending on the comparison between H and (a power of) κ.
There are two regimes describing the concentration of the superconductivity along Γ.
In a first regime (high H below $H_{c3}(\kappa)$) , the distribution of the superconductivity is displayed via a limiting function $E(\cdot)$; this limiting function is defined via a simplified Ginzburg-Landau type functional with a magnetic field vanishing along a line.
In another regime (lower H above $H_{c1}(\kappa)$) , the distribution of the superconductivity is displayed via a *known* limiting function $g(\cdot)$, which is defined via a simplified Ginzburg-Landau type functional with a *constant* magnetic field; the function $g(\cdot)$ displays the distribution of (bulk) superconductivity for samples submitted to a constant magnetic field (Sandier-Serfaty (2002), Fournais-Kashmar).

We recall that the Ginzburg-Landau functional is defined for $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ by,

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega e_{\kappa,H}(\psi, \mathbf{A})\, dx \tag{18}$$

where

$$e_{\kappa,H}(\psi, \mathbf{A}) := |(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + (\kappa H)^2|\operatorname{curl} \mathbf{A} - \beta|^2 .$$

We also assume that:

$$\Gamma \cap \partial\Omega \quad \text{is a } \textbf{finite} \text{ set}\,. \tag{19}$$

The assumptions on β and Γ (if non empty) force the function β to change sign. In physical terms, the set Γ splits the domain Ω into two parts $\Omega_1 = \{\beta(x) > 0\}$ and $\Omega_2 = \{\beta(x) < 0\}$ such that the magnetic field applied on Ω_1 is along the opposite direction of the magnetic field applied on Ω_2.

The ground state energy of the functional is,

$$\mathcal{E}_{min} = \inf\{\mathcal{G}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}\,. \quad (20)$$

We are interested in the asymptotics of the energy for κ large. The PHD thesis of K. Attar (2012-2014) is devoted to the case when $H \approx \kappa$.

Helffer-Kachmar (2014) consider the regime where H satisfies

$$H = \varsigma \kappa^2\,, \quad \varsigma \in \, (0, \infty)\,, \quad (21)$$

with ς being a function of κ satisfying $\varsigma \gg \kappa^{-1}$. This regime only appears when Γ is non empty.

Attar's results

They have been obtained recently by Attar in [3, 4]. The assumption on the strength of the magnetic field was $H \leq C\kappa$. In the regime of large κ, K. Attar has obtained in [3, 4] parallel results to those known for the constant magnetic field in Sandier-Serfaty [33].
However, it is proved in [3] that if

$$H = b\kappa \,, \tag{22}$$

and b is a constant, then when b is large enough, the energy and the superconducting density are concentrated near Γ with a length scale $\frac{1}{b}$.

Essentially, that is a consequence of the following theorem.

Theorem on the asymptotics of the energy

As $\kappa \to +\infty$, we have the following asymptotics

$$\mathcal{E}_{min} = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|\beta(x)|\right)\, dx + o\left(\kappa H \left|\ln \frac{\kappa}{H}\right| + 1\right), \tag{23}$$

which is valid under the relaxed assumption that

$$\Lambda_1 \kappa^{1/3} \leq H \leq \Lambda_2 \kappa\,, \tag{24}$$

Λ_1 and Λ_2 being positive constants.

Remarks

1. When $\beta > 0$ is constant, we have

$$\mathcal{E}_{min} = \kappa^2 |\Omega| \, g\left(\frac{H}{\kappa}\beta\right) + o\left(\kappa H \left|\ln \frac{\kappa}{H}\right| + 1\right) . \qquad (25)$$

   This is only good if $H < \kappa\beta$. This formula does not "see" the surface superconductivity ! For this we need another formula which was conjectured by Pan [29], Almog-Helffer [2], Fournais-Helffer, Fournais-Helffer-Persson [13], the final statement being achieved by Correggi-Rougerie in [6, 7].
2. The lower bound is unsatisfactory and probably partially technical. Can we replace the lower bound by $H_{c1}(\kappa)$, assuming that it has been defined properly).
3. The upperbound is satisfactory if β does not vanish (with Giorgi-Philipps statement in mind) and not satisfactory in the case when Γ is not empty.

Details on Pan's conjecture and its solution by Correggi-Rougerie

$\beta = 1$.

Theorem (see Pan [29])

For any $b \in \left]1, \Theta_0^{-1}\right[$, there exists a constant E_b, such that, for $H = \kappa b$, as $\kappa \to +\infty$,

$$\inf_{(\psi,\mathbf{A}) \in W^{1,2}(\Omega)\times W^{1,2}(\Omega;\mathbb{R}^2)} \mathcal{E}_{\kappa,H}[\psi,\mathbf{A}] = -\sqrt{\kappa H} E_b |\partial\Omega| + o(\kappa). \quad (26)$$

Pan's conjecture gives the value for E_b in relation with the family of $1D$-functionals

$$\mathcal{F}_{z,\lambda}(\phi) := \int_0^{+\infty} |\phi'(t)|^2 + (t-z)^2|\phi(t)|^2 + \frac{\lambda}{2}|\phi(t)|^4 - \lambda|\phi(t)|^2\, dt\,. \quad (27)$$

For $\lambda = b^{-1}$, we have to minimize over z the infimum of the functional over ϕ.

The function g hereafter called the bulk energy, appears in the analysis of the two and three dimensional Ginzburg-Landau functional with constant magnetic field Sandier-Serfaty (2002) [33], Fournais-Kachmar [14]. It is associated with some effective model energy. The function g will play a central role and its definition will be recalled later.

The bulk energy function

It is related to the periodic solutions of (5) and the Abrikosov energy (cf. Aftalion-Serfaty [1], Fournais-Kachmar [15]).
For $b \in (0, +\infty)$, $r > 0$, and $Q_r = (-r/2, r/2) \times (-r/2, r/2)$, we define the functional, for $u \in H^1(Q_r)$,

$$F_{b,Q_r}(u) = \int_{Q_r} \left(b|(\nabla - i\mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx\,. \qquad (28)$$

Here, $\mathbf{A}_0$ is the magnetic potential,

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)\,, \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2\,. \qquad (29)$$

We define the Dirichlet, Periodic, and Neumann ground state energies by

$$e_D(b,r) = \inf\{F_{b,Q_r}(u) \ : \ u \in H^1_0(Q_r)\}\,, \tag{30}$$

$$e_P(b,r) = \inf\{F_{b,Q_r}(u) \ : \ u \in H^1_{per}(Q_r)\}\,, \tag{31}$$

$$e_N(b,r) = \inf\{F_{b,Q_r}(u) \ : \ u \in H^1(Q_r)\}\,. \tag{32}$$

We can define $g(\cdot)$ as follows (cf. Sandier-Serfaty (2002) [33], Fournais-Kachmar [14], Attar (2013) [3])

$$\forall\ b > 0\,, \quad g(b) = \lim_{r\to\infty} \frac{e_D(b,r)}{|Q_r|} = \lim_{r\to\infty} \frac{e_N(b,r)}{|Q_r|}\,, \tag{33}$$

where $|Q_r| = r^2$ denotes the area of Q_r.
Moreover $g(\cdot)$ is a non decreasing continuous function s.t.

$$g(0) = -\frac{1}{2}\,, \quad g(b) < 0 \text{ when } b < 1\,, \text{ and } \quad g(b) = 0 \text{ when } b \geq 1\,. \tag{34}$$

One purpose is now to give a precise description of the aforementioned concentration of the order parameter and the energy when $\varsigma \gg 1$, thereby leading to the assumption in (21). The leading order term of the ground state energy in (20) is expressed in some regime via the new quantity (instead of $g(\cdot)$) $E(\cdot)$. The function $(0, \infty) \ni L \mapsto E(L)$ is a continuous function satisfying the following properties:

- $E(L)$ is defined via a reduced Ginzburg-Landau energy in the strip (this energy is introduced in (75)).
- $E(L) = 0$ iff $L \geq \lambda_0^{-3/2}$, where λ_0 is a universal constant defined as the bottom of the spectrum of a Montgomery operator.
- As $L \to 0_+$, the expected asymptotic behavior of $E(L)$ is like $L^{-4/3}$.

If time permits, this will be discussed more deeply at the end of this part.

The main result is:

Theorem (Helffer-Kachmar)

Suppose that β satisfies previous assumptions. Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying

$$\lim_{\kappa\to\infty} b(\kappa) = +\infty \quad \text{and} \quad \limsup_{\kappa\to\infty} \kappa^{-1} b(\kappa) < +\infty\,. \tag{35}$$

Suppose that

$$H = b(\kappa)\kappa\,.$$

Theorem continued

Then, as $\kappa \to +\infty$, the ground state energy in (20) satisfies:

1. If $b(\kappa) \gg \kappa^{1/2}$, then

$$\mathcal{E}_{min} = \kappa \left(\int_\Gamma \left(|\nabla \beta(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla\beta(x)| \frac{H}{\kappa^2} \right) ds(x) \right) + o\left(\frac{\kappa^3}{H} \right). \tag{36}$$

2. If $b(\kappa) \lesssim \kappa^{1/2}$, then

$$\mathcal{E}_{min} = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa} |\beta(x)| \right) dx + o\left(\frac{\kappa^3}{H} \right). \tag{37}$$

Case (2) (except the expression of the remainder) coincides with Attar's result.

First comment

We have seen that if H is larger than a critical value $H_{c_3}(\kappa)$, then the minimizers of the functional in (18) are trivial and the ground state energy is $\mathcal{E}_{min} = 0$. Furthermore, the value of $H_{c_3}(\kappa)$ as given in Pan-Kwek [28] admits, as $\kappa \to \infty$, the following asymptotics

$$H_{c_3}(\kappa) \sim c_0 \kappa^2 \,, \tag{38}$$

where c_0 is an explicit constant). As such, the assumption on the magnetic field in HK-Theorem is significant when
$b(\kappa)\kappa \leq H \leq M\kappa^2$ and $M \in (0, c_0]$ is a constant. Note also that this theorem gives a bridge between the situations studied by Attar in [3, 4] and Pan-Kwek in [28].

Remark: (The remainder terms in HK-Theorem

As long as the intensity of the external magnetic field satisfies $\kappa \ll H \leq M\kappa^2$ and $M \in (0, c_0)$, the remainder term appearing in HK-Theorem is of lower order compared with the principal term. $g(b)$ is bounded and vanishes when $b \geq 1$. Accordingly,

$$\kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|\beta(x)|\right)\, dx = \int_{\{|\beta(x)|<\frac{\kappa}{H}\}} g\left(\frac{H}{\kappa}|\beta(x)|\right)\, dx \approx \frac{\kappa^3}{H}\,.$$

One can see that,

$$\kappa \int_\Gamma \left(|\nabla\beta(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla\beta(x)|\frac{H}{\kappa^2}\right) \approx \frac{\kappa^3}{H}\,.$$

Hence they are of the same order

Remark (The two regimes in HK-Theorem)

HK-Theorem displays two regimes governing the behavior of the ground state energy. There is a small gap between the two regimes considered. Hence it is interesting to show that the two asymptotics match in this intermediate zone.
Hence it is interesting to inspect whether there exists a relationship between the limiting functions $E(\cdot)$ and $g(\cdot)$.

An asymptotic link between E and g

Helffer-Kachmar (2014) obtain the following relationship between the functions $E(\cdot)$ and $g(\cdot)$:

Matching Theorem

It holds,

$$E(L) = 2L^{-4/3} \int_0^1 g(b)\, db + o\big(L^{-4/3}\big) \quad \text{as } L \to 0_+\,.$$

Along the proof of HK-Theorem, we obtain:

Local HK-Theorem

Suppose β satisfies the previous assumptions. Let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be satisfying (35). Suppose $H = b(\kappa)\kappa$ and that $(\psi, \mathbf{A})$ is a minimizer of the functional in (18). Then, as $\kappa \to \infty$, it holds:

Estimate of the magnetic energy.

$$\kappa^2 H^2 \int_\Omega |\text{ curl } \mathbf{A} - \beta|^2 \, dx = \frac{\kappa^3}{H} o(1) \,.$$

So the induced magnetic field is close to the applied magnetic field.

Estimate of the local energy in D.

1. If $b(\kappa) \gg \kappa^{1/2}$, then

$$\mathcal{E}_0(\psi, \mathbf{A}; D) := \int_D \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx \tag{39}$$

$$= \kappa \left(\int_{D\cap\Gamma} \left(|\nabla\beta(x)| \frac{H}{\kappa^2} \right)^{1/3} E\left(|\nabla\beta(x)| \frac{H}{\kappa^2} \right) ds(x) \right) \tag{40}$$

$$+ \frac{\kappa^3}{H} o(1) .$$

2. If $1 \ll b(\kappa) \lesssim \kappa^{1/2}$, then

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \kappa^2 \int_D g\left(\tfrac{H}{\kappa}|\beta(x)|\right) dx + \tfrac{\kappa^3}{H} o(1) .$$

Concentration of the order parameter in D

.

1. If $b(\kappa) \gg \kappa^{1/2}$, then

$$\int_D |\psi(x)|^4\, dx = -\frac{2}{\kappa}\left(\int_{D\cap\Gamma}\left(|\nabla\beta(x)|\frac{H}{\kappa^2}\right)^{1/3} E\left(|\nabla\beta(x)|\frac{H}{\kappa^2}\right) ds(x$$

2. If $1 \ll b(\kappa) \lesssim \kappa^{1/2}$, then

$$\int_D |\psi(x)|^4\, dx = -\int_D g\left(\frac{H}{\kappa}|\beta(x)|\right) dx + \frac{\kappa}{H} o\,(1)\,.$$

This gives a weak localization result for the order parameter ψ in a neighborhood of Γ.

In the two regimes displayed in the local HK-Theorem, the mean term in the asymptotic expansions vanish when $D \cap \Gamma = \emptyset$. It could be interesting to improve the remainder terms, we will prove that the L^2-norm of ψ is concentrated near Γ, and that ψ exponentially decays as $\kappa \to \infty$, away from Γ.

Exponential decay outside the superconductivity region

We work under Attar-Assumptions (H is of the same order as κ).
We introduce $\omega(\epsilon) := \{x \in \bar{\Omega} \,|\, |\beta(x)| > \epsilon\}$.

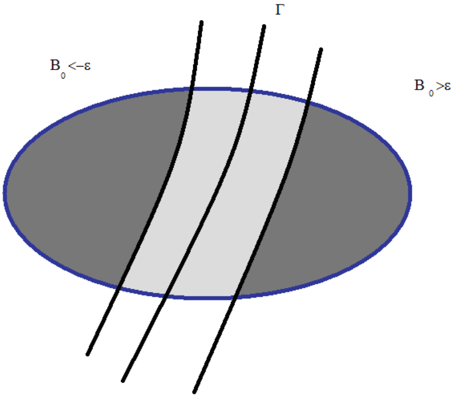


Figure: sample subject to a variable magnetic field vanishing along the curve Γ. Here $\beta = B_0$

Decay Theorem

Let β in $C^{0,\alpha}$, $\beta_0 = \max_{x\in\bar\Omega} |\beta(x)|$ and $\beta_1 = \max_{x\in\partial\Omega} |\beta(x)|$.
If $b > \beta_0^{-1}$ and $K \subset \omega\left(\frac{1}{b}\right)$ is an open set with $\overline{K} \subset \omega\left(\frac{1}{b}\right)$, there exist $\kappa_0 > 0$, $C > 0$ and $\alpha_0 > 0$ such that, if $\kappa \geq \kappa_0$ and $(\psi, \mathbf{A})_{\kappa,H}$ is a solution of (5) for $H = b\kappa$, then

$$\|\psi\|_{H^1(K)} \leq C\, e^{-\alpha_0\kappa}\,. \tag{41}$$

If $b > \left(\Theta_0\beta_1\right)^{-1}$, then the same holds when K satisfies

$$\overline{K} \subset \omega\left(\frac{1}{b}\right) \cup \left\{x \in \partial\Omega,\ \Theta_0|\beta(x)| > \frac{1}{b}\right\}.$$

The proof is based on Agmon like decay estimates.

Critical fields revisited

The identification of the critical *magnetic* fields is an important question regarding the functional in (18). This question has an early appearance in physics (see e.g. [17]) and was the subject of a vast mathematical literature in the past two decades. The two monographs [8, 32] contain an extensive review of many important results. We give a brief *informal* description of critical fields and highlight the importance of the case of a vanishing applied magnetic field.

Reminder: The constant field case

When β is a (non-zero) constant, three critical values are assigned to the magnetic field H, namely H_{c_1}, H_{c_2} and H_{c_3}. The behavior of minimizers (and critical points) of the functional in (18) changes as the parameter H (i.e. magnetic field) crosses the values H_{c_1}, H_{c_2} and H_{c_3}. The identification of these critical values is not easy, especially the value H_{c_2} which is still *loosely* defined.
Let us recall that a critical point $(\psi, \mathbf{A})$ of the functional in (18) is said to be *normal* if $\psi = 0$ everywhere. The critical field $H_{c_3}(\kappa)$ is then defined as the value at which the transition from *normal* to *non-normal* critical points takes place.

The identification of the critical value $H_{c_3}(\kappa)$ is strongly related to the spectral analysis of the magnetic Schrödinger operator with a constant magnetic field and Neumann boundary condition. Suppose that $\Omega \subset \mathbb{R}^2$ is connected, open, F a vector field satisfying $\mathrm{curl} F = \beta$, with β is constant and positive, and $\lambda(H\kappa F)$ the lowest eigenvalue of the magnetic Schrödinger operator

$$-\Delta_{\kappa HF} = -(\nabla - i\kappa HF)^2 \quad \text{in } L^2(\Omega)\,, \tag{42}$$

with Neumann boundary conditions. It has been proved that the function $t \mapsto \lambda^N(tF)$ is monotonic for large values of t, see [8] and the references therein.

The four fields are proved to be equal in [8]. the critical field H_{c_3} is the unique solution of:

$$\lambda^N(H_{c_3}(\kappa)\kappa F) = \kappa^2\,. \tag{43}$$

In this case, it was shown by Lu-Pan [24] that,

$$\lambda^N(H\kappa F) \sim (H\kappa)\beta\Theta_0\,, \text{ when } H\kappa \gg 1\,. \tag{44}$$

Further improvements of (44) are available, see [8] for the state of the art in 2009 and references therein.
As a consequence of (43) and (44), we get for κ sufficiently large,

$$H_{c_3}(\kappa) \sim \kappa/(\Theta_0\beta)\,. \tag{45}$$

The second critical field $H_{c_2}(\kappa)$ is usually defined as follows

$$H_{c_2}(\kappa) = \kappa/\beta\,. \tag{46}$$

Notice that this definition of H_{c_2} is asymptotically matching with the following definition[1],

$$\lambda^D(H_{c_2}(\kappa)\kappa F) = \kappa^2\,, \tag{47}$$

where λ^D is the first eigenvalue of the operator in (42), but with Dirichlet boundary condition.
Near $H_{c_2}(\kappa)$, a transition takes place between surface and bulk superconductivity. At the level of the energy, this transition is described in [15]. The bulk distribution of the superconductivity near H_{c_2} is computed in [22].

[1] Assuming the monotonicity of $t \mapsto \lambda^D(tF)$ for t large.

We recall that $\Theta_0 < 1$. Hence, as expected, $H_{c_2}(\kappa) < H_{c_3}(\kappa)$ for κ sufficiently large. For the identification of the critical field $H_{c_1}(\kappa)$, we refer to Sandier-Serfaty [32]. A natural question is to extend this discussion in the variable magnetic field case (i.e. β is non-constant).

The case of a non vanishing exterior magnetic field

Here we discuss the situation where $\beta(x) \neq 0$ everywhere in $\overline{\Omega}$. In this case, we recall that it is proved by Lu-Pan [25, Theorem 1] that,

$$\lambda(H\kappa F) \sim (H\kappa) \min \left(\inf_{x \in \overline{\Omega}} |\beta(x)|, \Theta_0 \inf_{x \in \partial\Omega} |\beta(x)| \right) , \qquad (48)$$

as $H\kappa \to \infty$.

Basically, this leads to consider two cases as follows.

Surface superconductivity

First, we assume that

$$\inf_{x \in \overline{\Omega}} |\beta(x)| > \Theta_0 \inf_{x \in \partial\Omega} |\beta(x)| . \qquad (49)$$

In this case, the phenomenon of surface superconductivity observed in the constant magnetic field case is preserved. More precisely, superconductivity starts to appear at the points where $(\beta)_{/\partial\Omega}$ is minimal. The critical value $H_{c_3}(\kappa)$ is still defined by (43). If the minima of $(\beta)_{/\partial\Omega}$ are non-degenerate, then the monotonicity of the eigenvalue $\lambda^N(t\,F)$ for large values of t is established by N. Raymond in [30, Section 6]. Consequently, we get when κ is sufficiently large,

$$H_{c_3}(\kappa) \sim \frac{\kappa}{\Theta_0 \inf_{x\in\partial\Omega} |\beta(x)|}\,. \tag{50}$$

Tentatively, one could think to define $H_{c_2}(\kappa)$ either by

$$H_{c_2}(\kappa) = \frac{\kappa}{\inf_{x\in\overline{\Omega}} |\beta(x)|}\,, \tag{51}$$

or by

$$\lambda^D(H_{c_2}(\kappa)\kappa F) = \kappa^2\,, \tag{52}$$

where λ^D is the first Dirichlet eigenvalue of the operator in (42).

Both formulas agree with their analogues in the constant magnetic field case (see (46) and (47)). Also, the values of $H_{c_2}(\kappa)$ given in (51) or (52) asymptotically match as $\kappa \to \infty$.

In order that the definition of $H_{c_2}(\kappa)$ in (52) be consistent, one should prove monotonicity of $t \mapsto \lambda^D(tF)$ for large values of t. This will ensure that (52) assigns a unique value of $H_{c_2}(\kappa)$. However, such a monotonicity is not proved yet. The definition in (51) was proposed in [8].

Interior onset of superconductivity Here we assume that

$$\inf_{x\in\overline{\Omega}} |\beta(x)| < \Theta_0 \inf_{x\in\partial\Omega} |\beta(x)| \,. \tag{53}$$

In this case, the onset of superconductivity near the surface of the domain disappears. If one decreases gradually the intensity of H from $+\infty$, then superconductivity will start to appear near the minima of $|\beta|$, i.e. inside a compact subset of Ω.

In this situation, we need not distinguish between the critical fields $H_{c_2}(\kappa)$ and $H_{c_3}(\kappa)$, since surface superconductivity is absent here. Consequently, we expect that,

$$H_{c_2}(\kappa) = H_{c_3}(\kappa) \sim \frac{\kappa}{\inf_{x\in\overline{\Omega}} |\beta(x)|}\,. \tag{54}$$

A partial justification of this fact can be done using the linearized Ginzburg-Landau equation near a normal solution. Actually, we may also define $H_{c_3}(\kappa)$ and $H_{c_2}(\kappa)$ as the values verifying (43) and (47). It should be noticed here that the vector field F satisfies $\text{curl } F = \beta$ and β cannot be constant.

Under Assumption (53), the known spectral asymptotics (which are actually the same in this case) of the Dirichlet and Neumann eigenvalues will lead us to the asymptotics given in the righthand side of (54). Under the additional assumption that $\inf_{x\in\overline{\Omega}} |\beta(x)|$ is attained at a unique minimum in Ω and that this minimum is non degenerate, a complete asymptotics of $\lambda^N(tF)$ can be given (see Helffer-Mohamed [21], Helffer-Kordyukov [19, 20], Raymond-Vu Ngoc [31]) and the monotonicity/strong diamagnetism property holds for large values of t (see Chapter 3 in [8]). Hence the definition of $H_{c_3}(\kappa)$ is clear in this case.

Besides the aforementioned linearized calculations, the results of [3] can be used to justify the equality of the critical fields $H_{c_2}(\kappa)$ and $H_{c_3}(\kappa)$ as well as their definition in (54).

Critical fields in the case of a vanishing exterior magnetic field

We come back the case when β vanishes along a curve, first considered in [28] and then in [3]. We assume that

$$|\beta| + |\nabla \beta| \neq 0 \text{ in } \overline{\Omega}\,. \tag{55}$$

At each point of $\beta^{-1}(0) \cap \Omega$, Pan-Kwek [28] introduce a reduced model (a Montgomery operator parameterized by the intensity of the magnetic field at this point) whose ground state energy, denoted by λ_0, captures the 'local' ground state energy of the Schrödinger operator in (42).

Similarly, at every point x of $\beta^{-1}(0) \cap \partial\Omega$, a toy operator is defined on $\mathbb{R}^2_+$ parameterized (up to unitary equivalence) by the intensity of $\beta(x)$ and the angle $\theta(x) \in [0, \pi/2)$ between the unit normal of the boundary and $\nabla\beta(x)$. The ground state energy of this toy operator is denoted by $\lambda_0(\mathbb{R}_+, \theta(x))$.
We recall that the leading order behavior of the ground state energy of the operator in (42) is

$$\lambda^N(H\kappa F) \sim (H\kappa)^{2/3}\, \alpha_1^{2/3}\,, \tag{56}$$

as $H\kappa \to +\infty$.

Here

$$\alpha_1 = \min\left(\lambda_0^{3/2} \min_{x\in\Gamma_{\rm blk}} |\nabla\beta(x)|\,,\ \min_{x\in\Gamma_{\rm bnd}} \lambda_0(\mathbb{R}_+,\theta(x))|\nabla\beta(x)|\right)\,, \tag{57}$$

$$\Gamma_{\rm blk} = \{x\in\Omega\ :\ \beta(x)=0\} \tag{58}$$

and

$$\Gamma_{\rm bnd} = \{x\in\partial\Omega\ :\ \beta(x)=0\}\,. \tag{59}$$

The critical value $H_{c_3}(\kappa)$ could tentatively be defined as the solution of the equation in (43). However, when β vanishes, monotonicity of $t \mapsto \lambda^N(tF)$ is not a direct application of Chapter 3 in [8] (see the discussion below). Nevertheless, for the various definitions of $H_{c_3}(\kappa)$ proposed in [28], one always get that, for large values of κ,

$$H_{c_3}(\kappa) \sim \frac{\kappa^2}{\alpha_1}\,. \tag{60}$$

Surface superconductivity (near H_{c_3}) is absent if

$$\lambda_0^{3/2} \min_{x \in \Gamma_{\mathrm{blk}}} |\nabla \beta(x)| < \min_{x \in \Gamma_{\mathrm{bnd}}} \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla \beta(x)| \,,$$

and in this case, we do not distinguish between H_{c_2} and H_{c_3}.
However, if

$$\lambda_0^{3/2} \min_{x \in \Gamma_{\mathrm{blk}}} |\nabla \beta(x)| > \min_{x \in \Gamma_{\mathrm{bnd}}} \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla \beta(x)| \,, \tag{61}$$

the phenomenon of surface superconductivity is observed in decreasing magnetic fields. Superconductivity will nucleate near the minima of the function

$$\Gamma_{\mathrm{bnd}} \ni x \mapsto \lambda_0(\mathbb{R}_+, \theta(x)) |\nabla \beta(x)| \,.$$

In this case, a natural definition of $H_{c_2}(\kappa)$ can be,

$$H_{c_2}(\kappa) := \frac{\kappa^2}{\alpha_2} \,, \tag{62}$$

for large values of κ , with

$$\alpha_2 = \lambda_0^{3/2} \min_{x \in \Gamma_{\mathrm{blk}}} |\nabla \beta(x)| \,.$$

The methods in [10] suggest that the monotonicity of the eigenvalue $\lambda^N(tF)$ for large values of t can be obtained in the case when (61) is satisfied. A necessary step is to find the second correction term in (56). The work in [26] is along this direction. Clearly, the condition in (24) is violated when the intensity H is comparable with the critical value $H_{c_3}(\kappa) \approx \kappa^2$, thereby preventing the application of the results of Attar [3].

The limiting problem: $E(L)$

We come back to the analysis of the limiting function $E(\cdot)$ function which arises as the limit of a certain simplified Ginzburg-Landau functional with a magnetic field vanishing along a line.

The Montgomery operator

Consider the self-adjoint operator in $L^2(\mathbb{R}^2)$

$$P = -\left(\partial_{x_1} - i\frac{x_2^2}{2}\right)^2 - \partial_{x_2}^2 \,. \tag{63}$$

The ground state energy

$$\lambda_0 = \inf \sigma(P) \tag{64}$$

of the operator P is described using the Montgomery operator as follows.
If $\tau \in \mathbb{R}$, let $\lambda^M(\tau)$ be the first eigenvalue of the Montgomery operator [27],

$$P(\tau) = -\frac{d^2}{dx_2^2} + \left(\frac{x_2^2}{2} + \tau\right)^2, \quad \text{in } L^2(\mathbb{R})\,. \tag{65}$$

Notice that the eigenvalue $\lambda^M(\tau)$ is positive, simple and has a unique positive eigenfunction φ^τ of L^2 norm 1. There exists a **unique** $\tau_0 \in \mathbb{R}$ such that

$$\lambda_0^M = \lambda^M(\tau_0)\,. \tag{66}$$

Hence $\lambda_0 > 0$. We write

$$\varphi_0 = \varphi^{\tau_0}$$

Moreover (see [18] and references therein) the minimum of λ^M at τ_0 is non-degenerate.

A one dimensional energy

Here $\lambda = \lambda^M$.
Let $b > 0$ and $\alpha \in \mathbb{R}$ and consider

$$\mathcal{E}^{1D}_{\alpha,b}(f) = \int_{-\infty}^{\infty} \left(|f'(t)|^2 + \left(\frac{t^2}{2} + \alpha\right)^2 |f|^2 - b\,|f(t)|^2 + \frac{b}{2}\,|f(t)|^4 \right) dt\,, \tag{67}$$

defined on $B^1(\mathbb{R}) = \{f \in H^1(\mathbb{R};\mathbb{R}) \ : \ t^2 f \in L^2(\mathbb{R})\}\,.$
Let $z_1(b)$ and $z_2(b)$ satisfying,

$$z_1(b) < \tau_0 < z_2(b)\,, \quad \lambda^{-1}\big([\tau_0, b)\big) = (z_1(b), z_2(b))\,. \tag{68}$$

Notice that, if $b < \lambda(0)$, then $z_2(b) < 0$.

Theorem

1. $\mathcal{E}^{1D}_{\alpha,b}$ has a non-trivial minimizer in $B^1(\mathbb{R})$ if and only if $\lambda^M(\alpha) < b$.
   Furthermore, there exists a non-trivial positive minimizer f_α and $\pm f_\alpha$ are the only real-valued minimizers.
2. Let
   $$\beta(\alpha, b) = \inf\{\mathcal{E}^{1D}_{\alpha,b}(f) \ : \ f \in B^1(\mathbb{R})\}\,. \quad (69)$$
   Then $\exists \alpha_0 \in (z_1(b), z_2(b))$ s.t.
   $$\beta(\alpha_0, b) = \inf_{\alpha\in\mathbb{R}} \beta(\alpha, b)\,. \quad (70)$$
3. If $b < \lambda^M(0)$, then $\alpha_0 < 0$.
4. $$\int_{-\infty}^{\infty} \left(\frac{t^2}{2} + \alpha_0\right) |f_{\alpha_0}(t)|^2\, dt = 0\,. \quad (71)$$

The proof of this theorem can be adapted from the analysis of Fournais-Helffer [8, Sec. 14.2] devoted to the functional

$$\mathcal{F}^{1D}_{\alpha,b}(f) = \int_0^{\infty} \left(|f'(t)|^2 + (t+\alpha)^2 \, |f(t)|^2 - b|f|^2 + \frac{b}{2}|f|^4 \right) \, dt \,. \tag{72}$$

We note that a minimizer of $\mathcal{E}^{1D}_{\alpha,b}$ satisfies the Euler-Lagrange equation:

$$-f''(t) + (\frac{t^2}{2} + \alpha)^2 f(t) - bf(t) + bf(t)^3 = 0 \,, \tag{73}$$

and that $f \in \mathcal{S}(\mathbb{R})$.
We also observe that the functional $\mathcal{E}^{1D}_{\alpha,b}$ has non-trivial minimizers if and only if $\alpha \in (z_1(b), z_2(b))$.

Reduced Ginzburg-Landau functional

Let $L > 0$, $R > 0$, $\mathcal{S}_R = (-R, R) \times \mathbb{R}$ and

$$\mathbf{A}(x) = \left(-\frac{x_2^2}{2}, 0\right), \quad \left(x = (x_1, x_2) \in \mathcal{S}_R = (-R, R) \times \mathbb{R}\right). \quad (74)$$

Consider the functional

$$\mathcal{E}_{L,R}(u) = \int_{\mathcal{S}_R} \left(|(\nabla - i\mathbf{A})u|^2 - L^{-2/3}|u|^2 + \frac{L^{-2/3}}{2}|u|^4 \right) dx \,, \quad (75)$$

and the ground state energy $e(L; R)$ of $\mathcal{E}_{L,R}$

$$\inf\{\mathcal{E}_{L,R}(u) \,:\, (\nabla - i\mathbf{A})u \in L^2(\mathcal{S}_R)\,,\ u \in L^2(\mathcal{S}_R)\,,\ \text{and } u = 0 \text{ on } \partial\mathcal{S}_R\}\,. \quad (76)$$

Following the analysis in [29] and [16, Theorem 3.6], we can prove that the functional in (75) has a minimizer. The value of the ground state energy defined in (76) is related to the eigenvalue λ_0 in (66). We will find that the energy vanishes when $L \geq \lambda_0^{-3/2}$, and we will give a rough estimate of the energy when $L \to \lambda_0^{-3/2}$.

Proposition

For $R > 0$, $L > 0$,

1. If $L \geq \lambda_0^{-3/2}$, then $e(L;R) = 0$.
2. There exist positive C_1, C_2 and C_3 such that, if $L < \lambda_0^{-3/2}$ and $R > 0$, then

$$-C_1\, L^{-4/3} R \leq \frac{e(L;R)}{(1-\lambda_0 L^{2/3})} \leq -C_2\, L^{-2/3} R + \frac{C_3}{R}\,. \qquad (77)$$

One can prove that:

1. If $L \geq \lambda_0^{-2/3}$, then $\varphi_{L,R} = 0$ is the minimizer of the functional in (75) realizing the ground state energy in (76).
2. If $L \leq \lambda_0^{-2/3}$, every minimizer $\varphi_{L,R}$ satisfies,

$$\begin{aligned} &\int_{\mathcal{S}_R} |(\nabla - i\mathbf{A})\varphi_{L,R}(x)|^2 \, dx \leq CL^{-4/3} \,, \\ &\int_{\mathcal{S}_R} |\varphi_{L,R}(x)|^2 \, dx \leq CL^{-2/3} R \,. \end{aligned} \tag{78}$$

Notice that the energy $\mathcal{E}_{L,R}(u)$ in (75) is invariant under translation along the x_1-axis. This allows us to follow the approach in [14, 29] and obtain that the limit of $\frac{e(L;R)}{R}$ as $R \to \infty$ exists. The precise statement is:

Theorem

Given $L > 0$, there exists $E(L) \leq 0$ such that,

$$\lim_{R \to \infty} \frac{e(L;R)}{2R} = E(L)\,.$$

$(0,\infty) \ni L \mapsto E(L) \in (-\infty, 0]$ is continuous, monotone increasing and

$$E(L) = 0 \quad \text{if and only if } L \geq \lambda_0^{-3/2}\,.$$

Furthermore,

$$\forall\ R > 0\,, \forall L > 0\,, \quad E(L) \leq \frac{e(L;R)}{2R}\,, \tag{79}$$

and

$$\forall\ R \geq 2\,, \forall L > 0\,, \quad \frac{e(L;R)}{2R} \leq E(L) + C\left(1 + L^{-2/3}\right) R^{-2/3}\,. \tag{80}$$

It would be desirable to establish a simpler expression of $E(L)$ when $L \in (\lambda^M(0)^{-\frac{3}{2}}, \lambda_0^{-\frac{3}{2}})$:

Conjecture

If

$$\lambda_0 < L^{-2/3} < \lambda^M(0)\,, \tag{81}$$

then

$$E(L) = E^{1\mathrm{D}}(L^{-2/3})\,.$$

Here, for $b > 0$, $E^{1\mathrm{D}}(b) = \beta(\alpha_0, b)$ and $\beta(\alpha_0, b)$ is defined in (70).

In the case of a constant magnetic field, a similar statement to this Conjecture has been conjectured by X. Pan (2002) in [29]. As already mentioned, partial affirmative answers were given in Almog-Helffer [2] and Fournais-Helffer-Persson[13]. The conjecture has been finally proved recently by Correggi-Rougerie in [6, 7]. Their methods do not work in the case considered here.

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