Decay of Correlations in 2d Quantum Systems

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The results presented are part of a work-in-progress paper by C.B., J. Fröhlich and D. Ueltschi.

Phase transitions and symmetries

- Symmetry: group of transformation that leaves the system unaltered.
- Phase transition due to symmetry breaking: if $T < T_c$ the system favours an ordered state.
- Continuous symmetries (e.g. $\mathbf{U(1)}$) VS Discrete symmetries (e.g. $\mathbb{Z}_2$)

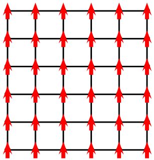


Figure: $T < T_c$

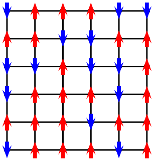


Figure: $T > T_c$

What about dimensions?

Mermin Wagner Theorem: no spontaneous breaking of a **continuous** symmetry can happen at $d \leq 2$ if $T > 0$.

N.B. This statement does not apply to discrete symmetries (e.g. $\mathbb{Z}_2$ symmetry in the ferromagnetic Ising model).

Phase transitions and correlation functions

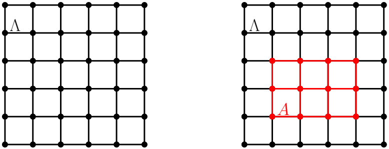
- Study of the decay rate of the relevant correlation functions to study the absence or presence of symmetry breaking.
- Expected decay rate in $d = 2$: **power law** (no spontaneous symmetry breaking!).

A general setting

Our aim is to find a general setting for the relevant correlation functions to decay algebraically, with a focus on **quantum models on a 2d lattice**. We will be interested in systems with a **U(1) symmetry**.

We will show how this setting is fulfilled by a great variety of well studied quantum systems.

A general setting: lattice and operators



- Finite lattice $(\Lambda, \mathcal{E})$ with **bounded** perimeter constant γ:

$$\gamma = \max_{x \in \Lambda} \max_{\ell \in \mathbb{N}} \frac{1}{\ell} |\{y \in \Lambda | d(x, y) = \ell\}|$$

- Maximal degree of the lattice:

$$\Delta = \max_{x \in \Lambda} |\{y \in \Lambda | d(x, y) = 1\}|$$

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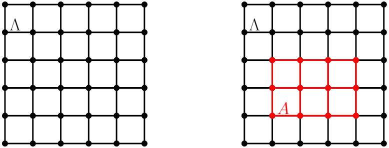

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- Hilbert space $\mathcal{H}_\Lambda$ (e.g. quantum spin systems: $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2\mathcal{S}+1}$).
- Space of bounded operators on $\mathcal{H}_\Lambda$: $\mathcal{B}_\Lambda$.

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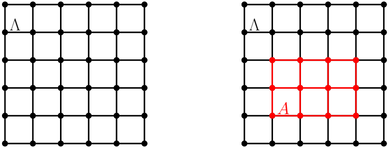

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- Hilbert space $\mathcal{H}_\Lambda$ (e.g. quantum spin systems: $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2\mathcal{S}+1}$).
- Space of bounded operators on $\mathcal{H}_\Lambda$: $\mathcal{B}_\Lambda$.
- Local operators: $\mathcal{B}_A \subset \mathcal{B}_\Lambda$ for $A \subset \Lambda$ (e.g. quantum spin systems: given an operator $\mathcal{O}_A$ defined on $\otimes_{x\in A}\mathbb{C}^{2\mathcal{S}+1}$ it can be identified with $\mathcal{O}_\Lambda = \mathcal{O}_A \otimes \mathbf{1}_{\Lambda\setminus A}$)

A general setting: interactions and Hamiltonian

- Interaction Φ: collection of local operators $\{\Phi_A\}_{A\subset\Lambda}$ with $\Phi_A \in \mathcal{B}_A$ for $A \subset \Lambda$. **Finite norm**:

$$\|\Phi\| = \sum_{r\geq 1} \sup_{x\in\Lambda} \sum_{\substack{A\ni x \\ \mathrm{diam}(A)=r}} r\|\Phi_A\|_\infty |A|^4$$

- Hamiltonian defined through the interaction:

$$H_\Lambda = \sum_{A\subset\Lambda} \Phi_A$$

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- Gibbs state:

$$\langle a\rangle_\Lambda = \frac{1}{Z_\Lambda}\text{Tr}ae^{-\beta H_\Lambda} \qquad Z_\Lambda = Tre^{-\beta H_\Lambda}$$

for any observable a on $\mathcal{H}_\Lambda$

A general setting: U(1) symmetry

- $(S_x)_{x\in\Lambda}$ local operators such that $\|S_x\|_\infty = S \ \forall x \in \Lambda$ and:

$$\left[\Phi_A, \sum_{x\in A} S_x\right] = 0$$

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.

- Correlation function: for any x, y in Λ local operator $O_{xy} \in \mathcal{B}_{\{x,y\}}$ with $\|O_{xy}\|_\infty$ not depending on x, y and such that

$$[S_x, O_{xy}] = -[S_y, O_{xy}] = cO_{xy}$$

for some $c \in \mathbb{R}$, for any $x, y \in \Lambda$.

Decay of correlations

Theorem (C. B., J. Fröhlich, D. Ueltschi)

Let $\{\Phi_A\}_{A\subset\Lambda}$, $(S_x)_{x\in\Lambda}$, O_{xy} *be as just described. Then:*

$$\begin{aligned} |\langle O_{xy}\rangle| &\leq \|O_{xy}\|_\infty e^{-\xi(x,y)} \\ \xi(x,y) &= \sup_{\substack{\{\theta_z\}_{z\in\Lambda}\in\mathbb{R}^\Lambda \\ \theta_y=0}} \left(\theta_x - 2\beta S^2 \|\Phi\| \sum_{\langle u,v\rangle\in\mathcal{E}} (\theta_u-\theta_v)^2 \right). \end{aligned}$$

Moreover:

$$\xi(x,y) \geq \frac{c^2}{16\beta\gamma^2\|\Phi\|S^2} \log(d(x,y)+1) - C(\beta,\|\Phi\|,S,\gamma).$$

What does it mean?

The theorem implies the following algebraic decay for the correlation function:

$$|\langle O_{xy} \rangle| \leq \frac{const.}{(d(x,y)+1)^{\frac{c^2}{16\beta\gamma^2\|\Phi\|S^2}}}$$

The power is **proportional to** $\frac{1}{\beta}$ and is **uniform** with respect to the size of the lattice Λ.

The proof relies on the so called Complex Rotation Method.

- O. A. McBryan, T. Spencer, *On the decay of correlations in SO(n)-symmetric ferromagnets. Commun. Math. Phys. 53, 299-302 (1977).*
- *T. Koma, H. Tasaki, Decay of Superconducting and Magnetic Correlations in One- and Two-Dimensional Hubbard Models. Phys. Rev. Lett. 68, 3248 (1992).*
- *J. Fröhlich, D. Ueltschi, Some properties of correlations of quantum lattice systems in thermal equilibrium. Math. Phys. 56, 053302 (2015).*

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Ex. 1 - SU(2) invariant spin system

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2\mathcal{S}+1}$$

$$H_\Lambda = -\sum_{\langle x,y\rangle\in\mathcal{E}}\sum_{k=1}^{2\mathcal{S}} c_k(\vec{\mathcal{S}}_x\cdot\vec{\mathcal{S}}_y)^k$$

with $\mathcal{S}\in\frac{1}{2}\mathbb{N}$, $\vec{\mathcal{S}}=(\mathcal{S}^1,\mathcal{S}^2,\mathcal{S}^3)$ spin-$\mathcal{S}$ matrices and $\mathcal{S}_x^i=\mathcal{S}^i\otimes_{\Lambda\backslash x}\mathbf{1}$

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Theorem (C.B., J. Fröhlich, D. Ueltschi)

For $i = 1,2,3$, we have

$$|\langle\mathcal{S}^i_x\mathcal{S}^i_y\rangle| \le C\,(d(x,y)+1)^{-c_1/\beta},$$

where C is a constant depending only on $\mathcal{S},\beta$, γ, Δ and c_1 is given by

$$c_1 = \left(256\,\mathcal{S}^2\,\gamma^2\,\Delta\left\|\sum_{k=1}^{2\mathcal{S}} c_k\left(\vec{\mathcal{S}}_x\cdot\vec{\mathcal{S}}_y\right)\right\|_\infty\right)^{-1}.$$

Ex. 2 - Hubbard Model

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\text{span}\{0,\uparrow,\downarrow,\uparrow\downarrow\} \simeq \otimes_{x\in\Lambda}\mathbb{C}^4$$
$$H_\Lambda = -\sum_{x,y\in\Lambda}\sum_{\sigma=\uparrow,\downarrow} t_{xy} c^\dagger_{\sigma,x} c_{\sigma,y} + V\left(\{n_{\sigma,x}\}\right)$$

where $V\left(\{n_{\sigma,x}\}\right)$ is a generic potential depending only on the total number of particles of any possible spin per site.

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where $V\left(\{n_{\sigma,x}\}\right)$ is a generic potential depending only on the total number of particles of any possible spin per site.
We need the following to be finite:

$$\tau = \sum_{r\geq 1}\max_{x\in\Lambda}\sum_{y\in\Lambda,d(x,y)=r} r|t_{xy}|.$$

Theorem (C.B., J. Fröhlich, D.Ueltschi)

The following bounds hold:

$$|\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle| \leq C_1 \left(d(x,y)+1\right)^{-c_1/\beta},$$
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$$|\langle c^\dagger_{\sigma,x} c_{\sigma,y}\rangle| \leq C_3 \left(d(x,y)+1\right)^{-c_3/\beta},$$

where $\sigma \in \{\uparrow,\downarrow\}$ *in the last line. Here* C_1, C_2, C_3 *depend only on* β, t_{xy}, Δ, *and* γ, *and* c_1, c_2, c_3 *are given by*

$$c_1 = (16\sqrt{2}\gamma^2\tau)^{-1}, \quad c_2 = (256\sqrt{2}\gamma^2\tau)^{-1}, \quad c_3 = (1024\sqrt{2}\gamma^2\tau)^{-1}.$$

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\begin{aligned}
|\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle| &\le C_1\,(d(x,y)+1)^{-c_1/\beta}\,,\\
|\langle c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} c_{\uparrow,y} c_{\downarrow,y}\rangle| &\le C_2\,(d(x,y)+1)^{-c_2/\beta}\,,\\
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$$

N.B. $\|\Phi\| < \infty$ only if $\tau < \infty$, which happens if $t_{xy} \sim d(x,y)^{-\alpha}$ with $\alpha > 3$. This is an improvement on **Koma and Tasaki (1992)** and **Macris and Ruiz (1994)**.

Other examples

- t-J model: same correlations as for the Hubbard model
- Random loops model: probability that two sites belong to the same loop

We have shown:

- Algebraic decay of correlation functions in U(1)-symmetric 2d quantum systems.
- Many possible relevant physical applications.

Future perspective:

- Bosonic systems.

Thank you!

Proof of the Theorem - 1

Let $\{\theta_y\}_{y\in\Lambda} \in \mathbb{R}^\Lambda$ be arbitrary real numbers for each site of the lattice. We define

$$R = \prod_{y\in\Lambda} e^{\theta_y S_y}.$$

To each subset $A \subset \Lambda$, we designate a special site $x_0 = x_0(A)$ in A. We get:

$$\begin{aligned} R H_\Lambda R^{-1} &= \sum_{A\subset\Lambda} e^{\sum_{y\in A}(\theta_y - \theta_{x_0(A)})S_y} \Phi_A e^{-\sum_{y\in A}(\theta_y - \theta_{x_0(A)})S_y} \\ &= e^{T_A}\Phi_A e^{-T_A}, \end{aligned}$$

where

$$T_A = \sum_{y\in A}(\theta_y - \theta_{x_0})S_y.$$

Proof of the Theorem - 2

Recall that $\mathrm{ad}_a(b) = [a, b]$. By multicommutator expansion:

$$RH_\Lambda R^{-1} = \sum_{A\subset\Lambda} \Phi_A + \sum_{j\geq 1}\sum_{A\subset\Lambda} \frac{1}{j!}\,\mathrm{ad}^j_{T_A}(\Phi_A)$$
$$= H_\Lambda + B + C,$$

where

$$B = \sum_{j\geq 1}\sum_{A\subset\Lambda} \frac{1}{(2j-1)!}\mathrm{ad}^{2j-1}_{T_A}(\Phi_A)$$

and

$$C = \sum_{j\geq 1}\sum_{A\subset\Lambda} \frac{1}{(2j)!}\mathrm{ad}^{2j}_{T_A}(\Phi_A).$$

B contains the odd terms of the multicommutator expansion and is thus anti-hermitian; C contains the even terms and is thus hermitian. Moreover;

$$\mathrm{e}^{\theta_x S_x + \theta_y S_y} O_{xy} \mathrm{e}^{-\theta_x S_x - \theta_y S_y} = \mathrm{e}^{c(\theta_x - \theta_y)} O_{xy}.$$

Proof of the Theorem - 3

We now apply the Trotter formula and the Hölder inequality for matrices.

$$\begin{aligned} \left|\mathrm{Tr}\left[O_{0x}\mathrm{e}^{-\beta H_\Lambda}\right]\right| &= \left|\mathrm{Tr}\left[RO_{0x}R^{-1}\mathrm{e}^{-\beta RH_\Lambda R^{-1}}\right]\right| \\ &= \mathrm{e}^{c(\theta_0-\theta_x)}\left|\mathrm{Tr}\left[O_{0x}\mathrm{e}^{-\beta H_\Lambda-\beta B-\beta C}\right]\right| \\ &\leq \mathrm{e}^{c(\theta_0-\theta_x)}\lim_{n\to\infty}\left|\mathrm{Tr}\left[O_{0x}\left(\mathrm{e}^{-\frac{\beta}{n}H_\Lambda}\mathrm{e}^{-\frac{\beta}{n}B}\mathrm{e}^{-\frac{\beta}{n}C}\right)^n\right]\right| \\ &\leq \mathrm{e}^{c(\theta_0-\theta_x)}\lim_{n\to\infty}\|O_{0x}\|_\infty\|\,\mathrm{e}^{-\frac{\beta}{n}H_\Lambda}\|_n^n\,\|\mathrm{e}^{-\frac{\beta}{n}B}\|_\infty^n\,\|\mathrm{e}^{-\frac{\beta}{n}C}\|_\infty^n. \end{aligned}$$

Notice that $\|\mathrm{e}^{-\frac{\beta}{n}B}\|_\infty = 1$ because B is anti-hermitian. Moreover, we have $\|\mathrm{e}^{-\frac{\beta}{n}H_\Lambda}\|_n^n = \mathrm{Tr}\left[\mathrm{e}^{-\beta H_\Lambda}\right]$, so that

$$|\langle O_{0x}\rangle| \leq \mathrm{e}^{c(\theta_x-\theta_y)}\|O_{0x}\|\mathrm{e}^{\beta\|C\|}.$$

Proof of the Theorem - 4

$$\|C\| \leq \sum_{j\geq 1}\sum_{A\subset\Lambda} \frac{1}{(2j)!}\|T_A\|^{2j}\|\Phi_A\|$$

$$\leq \sum_{j\geq 1}\sum_{A\subset\Lambda} \frac{(2)^{2j}}{(2j)!}\|S\|^{2j}\left(\sum_{x,y\in A}|\theta_x-\theta_y|\right)^{2j}\|\Phi_A\|.$$

From now on, we will suppose that the fourth-power term (that is $j=2$) is negligible, and ignore it. **Problem: how to handle the difference between angles at different sites**. For any $y, z \in A, A \subset \Lambda$ we choose a path $\Gamma_{y,z}$: a set of edges linking the two sites (with minimal length). If we denote $\langle u, v\rangle$ a couple of sites u and v which form an edge, we find:

$$|\theta_y - \theta_z| = \Big|\sum_{\langle u,v\rangle\in\Gamma_{y,z}}(\theta_u-\theta_v)\Big|.$$

In this way, the difference between the phases at different and possibly distant sites has been decomposed into the sum of differences between nearest-neighbours sites along a path connecting the two.

Proof of the Theorem -5

Notice that, given a sequence of N real numbers, a_n, $n \in \{1, \dots, N\}$ the following relation holds:

$$\left(\sum_{n=1}^{N} a_n\right)^2 \leq N \sum_{n=1}^{N} a_n^2$$

$$\begin{aligned}
\|C\| &\leq \sum_{A\subset\Lambda} 2\|S\|^2 \|\Phi_A\| \, |A|^2 \operatorname{diam}(A) \sum_{y,z\in A} \sum_{\langle u,v\rangle\in\Gamma_{y,z}} |\theta_u - \theta_v|^2 \\
&= \sum_{A\subset\Lambda} \sum_{y,z\in\Lambda} \sum_{\langle u,v\rangle\in\mathcal{E}} \mathbb{1}_{y,z\in A} \mathbb{1}_{\langle u,v\rangle\in\Gamma_{y,z}} 2\|S\|^2 \|\Phi_A\| \, |A|^2 \operatorname{diam}(A) \, |\theta_u - \theta_v|^2 \\
&\leq \sum_{A\subset\Lambda} \sum_{y,z\in\Lambda} \sum_{\langle u,v\rangle\in\mathcal{E}} \mathbb{1}_{y,z\in A} \mathbb{1}_{d(u,y)\leq\operatorname{diam}(A)} 2\|S\|^2 \|\Phi_A\| \, |A|^2 \operatorname{diam}(A) \, |\theta_u - \theta_v|^2 \\
&\leq \sum_{\langle u,v\rangle\in\mathcal{E}} |\theta_u - \theta_v|^2 \, 2\|S\|^2 \sum_{r\geq 1} \sup_{y\in\Lambda} \sum_{\substack{A\ni y \\ \operatorname{diam}(A)=r}} \|\Phi_A\| |A|^4 \operatorname{diam}(A) \\
&= \sum_{\langle u,v\rangle\in\mathcal{E}} |\theta_u - \theta_v|^2 \, 2\|S\|^2 \|\Phi\|
\end{aligned}$$

Proof of the Theorem - 6

We have proved:

$$|\langle O_{xy}\rangle| \leq \|O_{xy}\|_\infty e^{-\xi(x,y)}$$

The logarithmic decay of $\xi(x,y)$ is proved through a Taylor expansion, using the following Ansatz:

$$\theta_z = \begin{cases} \mathcal{K} \log\left(\frac{d(x,y)+1}{d(x,z)+1}\right) & \text{if } d(x,z) \leq d(x,y) \\ 0 & \text{otherwise} \end{cases}$$

for some $\mathcal{K} \in \mathbb{R}$.

Example - t-J Model

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\text{span}\{0,\uparrow,\downarrow,\uparrow\downarrow\} \simeq \otimes_{x\in\Lambda}\mathbb{C}^4$$

$$H_\Lambda = -t\sum_{\langle x,y\rangle\in\mathcal{E}}\sum_{\sigma=\uparrow,\downarrow} c^\dagger_{\sigma,x}c_{\sigma,y} + J\sum_{\langle x,y\rangle\in\mathcal{E}}\left(\vec{\mathcal{S}}_x\cdot\vec{\mathcal{S}}_y - \frac{1}{4}n_x n_y\right)$$

with $t, J \in \mathbb{R}$ and $\mathcal{S}^i_x = \frac{1}{2}\sum_{\mu,\nu=\uparrow,\downarrow} c^\dagger_{\mu,x}\sigma^i_{\mu,\nu}c_{\nu,x}$ (here, σ^i are the Pauli matrices).

Theorem (C.B., J. Fröhlich, D. Ueltschi)

The following bounds hold:

$$|\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle| \leq C_1 (d(x,y)+1)^{-c_1/\beta},$$
$$|\langle c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} c_{\uparrow,y} c_{\downarrow,y}\rangle| \leq C_2 (d(x,y)+1)^{-c_2/\beta},$$
$$|\langle c^\dagger_{\sigma,x} c_{\sigma,y}\rangle| \leq C_3 (d(x,y)+1)^{-c_3/\beta}$$

where $\sigma \in \{\uparrow,\downarrow\}$ in the last line. Here C_1, C_2, C_3 depend only on β, t, J, Δ and γ, and c_1, c_2, c_3 are given by

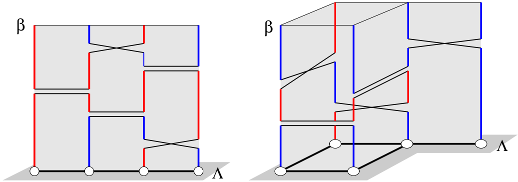
$$c_1 = \left(16\Delta\gamma^2 \left\| -t \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{\sigma,x} c_{\sigma,y} + \left(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \frac{1}{4} n_x n_y\right)\right\|_\infty\right)^{-1},$$

$$c_2 = \left(256\Delta\gamma^2 \left\| -t \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{\sigma,x} c_{\sigma,y} + J\left(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \frac{1}{4} n_x n_y\right)\right\|_\infty\right)^{-1},$$

$$c_3 = \left(1024\Delta\gamma^2 \left\| -t \sum_{\sigma=\uparrow,\downarrow} c^\dagger_{\sigma,x} c_{\sigma,y} + J\left(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \frac{1}{4} n_x n_y\right)\right\|_\infty\right)^{-1}.$$

Example - Random Loops Model

- To each edge of the lattice $(\Lambda, \mathcal{E})$ attach an interval $[0, \beta]$ with periodic boundary conditions.
- Define a Poisson point process with two types of events: crosses (intensity u) and bars (intensity $1-u$), $u \in [0,1]$.



- ρ_u probability measure, ω realisation, $\mathcal{L}(\omega)$ set of loops in the realisation ω.
- Partition function:

$$Z_\Lambda^\theta = \int_\Omega \theta^{|\mathcal{L}(\omega)|} d\rho_u(\omega), \qquad \theta = 2, 3, \ldots$$

- Relevant measure: $\qquad (Z_\Lambda^\theta)^{-1} \theta^{|\mathcal{L}(\omega)|} d\rho_u(\omega)$

Picture from D. Ueltschi, *Random loop representations for quantum spin systems*. J. Math. Phys. 54, (2013).

Theorem (C. Benassi, J. Fröhlich, D. Ueltschi)

Let $\theta = 2, 3, 4, \ldots$ *and* $u \in [0, 1]$*. Then we have*

$$\mathbb{P}(x \leftrightarrow y) \leq C\,(d(x, y) + 1)^{-c_1/\beta}$$

where C *and* c_1 *are a constants depending only on* θ, β, γ, u, Δ.

The proof is based on the fact that this random loops model can be mapped into a U(1)-symmetric spin-$\frac{\theta-1}{2}$ model.