Classical and Quantum parts of the quantum dynamics

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Time evolution of a close system

- Quantum system = Hilbert space $\mathcal{H} = \mathbb{C}^N$;
- We write $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear maps on $\mathcal{H}$;
- The Hamiltonian of the system is given by a selfadjoint operator $H \in \mathcal{B}(\mathcal{H})$.

Schrödinger Equation:

The time evolution is given by the equation:

$$dU_t = -itH\ U_t dt$$
$$U_0 = I_{\mathcal{H}}$$

The solution $(U_t)_{t\in\mathbb{R}}$ is a one-parameter group of unitary operators on $\mathcal{H}$:

- U_t is a unitary operator on $\mathcal{H}$ for all $t \in \mathbb{R}$;
- $U_{t+s} = U_t U_s$ for all $s, t \in \mathbb{R}$, and $U_0 = I_{\mathcal{H}}$;
- $t \mapsto U_t$ is strongly continuous.

Main problem:

How to model an open system?

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Markovian description in continuous time

- Let $(X_t)_{t\geq 0}$ be a stochastic process on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with value in $\mathbb{R}^d$;
- We can try to obtain a unitary solution to the following stochastic equation:

$$\text{1-dimensional process:} \qquad dU_t = -\left(iH + A\right) U_t dt + B U_t dX_t, \quad U_0 = I_{\mathcal{H}},$$

$$d\text{-dimensional process:} \qquad dU_t = -\left(iH + A\right) U_t dt + \sum_{k=1}^{d} B_k U_t dX_t^k,$$

  where $A, B \in \mathcal{B}(\mathcal{H})$.
- $A, B, B_k \in \mathcal{B}(\mathcal{H})$, have to be **chosen** with the process (X_t) so that there exists a **unique unitary solution**.
- Main point of the talk: (X_t) has to be a **Brownian process** or a **Poisson process**. Then we also know the form of A and B.

Why it is a good method:

- It is a simple way to model an open system for which we do not need a new theory;
- It allows us to use the different tools from stochastic calculus.

Why it is not enough:

- We only get quantum open dynamics with a classical environment!
- We do not get an explicit construction of the environment.

Goal

Integrate this framework into a quantum framework. The main tool is the **Quantum Stochastic Calculus** developed by Hudson and Parthasarathy in the 80s.

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Two examples: Brownian and Poisson processes

The case of a d-dimensional Brownian process $(B_t^1, \ldots, B_t^d)$:

$$A = \frac{1}{2}\sum_{k=1}^{d} L_k^2, \qquad L_k \in \mathcal{B}(\mathcal{H}), \quad L_k = L_k^*$$

$$B_k = L_k$$

The following equation admits a unique unitary solution:

$$dU_t = -\left(iH + \frac{1}{2}\sum_{k=1}^{d} L_k^2\right) U_t dt + \sum_{k=1}^{d} L_k U_t dB_t^k.$$

The case of a d-dimensional Poisson process of intensity $(N_t^1, \ldots, N_t^d)$, each coordinate of intensity ρ_k:

$$A = \frac{1}{2}\sum_{k=1}^{d} \rho_k^2(2I_{\mathcal{H}} - S_k - S_k^*), \qquad S_k \in \mathcal{B}(\mathcal{H}), \quad S_k S_k^* = I_{\mathcal{H}}$$

$$B_k = \rho_k(S_k - I_{\mathcal{H}})$$

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Random variable and their multiplication operators

Question: how can we integrate the classical setting inside the quantum one?

- Let X be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the law of X;
- Remark that we can recover all the information we want about X with the functionals:

$$f \mapsto \mathbb{E}[f(X)], \qquad f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$$

  (moments, characteristic functions, etc...)

- These functionals can be written on the Hilbert space level: take $\mathcal{K} = L^2(\mathbb{P})$ with scalar product

$$\langle f, g \rangle = \int_\Omega \overline{f} g \; d\mathbb{P}$$

  Then we have

$$\mathbb{E}[f(X)] = \langle \mathbb{1}, M_f \mathbb{1} \rangle,$$

  where $\mathbb{1}$ is the constant function equal to 1 and M_f is the operator of multiplication by f:

$$M_f g = fg, \qquad f \in L^\infty(\mathbb{P}),\; g \in L^2(\mathbb{P}).$$

- Conclusion: **We can replace the process X_t by its multiplication operator**

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The multiplication operator by the Brownian motion

- With the previous identification, it is possible to "identify" the Brownian motion with its multiplication operator:

$$M_{B_t^k} f = B_t^k f, \qquad f \in L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

- Then we can formally write the stochastic Schrödinger Equation with a Brownian noise as:

$$dU_t = -(iH + \frac{1}{2}\sum_{i=1}^{d} L_k^2) U_t dt + \sum_{i=1}^{d} L_k U_t \; M_{dB_t^k}, \qquad U_0 = I_{\mathcal{H}}$$

- It is no longer a **stochastic differential equation**. It is now an equation on the operator level, where there is nothing random! U_t **is now a unitary operator on** $\mathcal{H} \otimes L^2(\Omega, \mathcal{F}_t, \mathbb{P})$.
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The Probabilist Fock space

Definition

*The d-**multiple probabilist Fock space** is define as:*

$$\Phi(\mathbb{C}^d) = \mathcal{F}_B(L^2(\mathbb{R}^+, \mathbb{C}^d)) = \bigoplus_{n\geq 0} L^2(\mathbb{R}^+, \mathbb{C}^d)^{\vee n} \approx L^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$$

- On this space, we consider the usual creation, annihilation and number operators:

$$a_k^0(t) = a^*(\mathbb{1}_{[0,t]}|e_k\rangle), \quad a_0^k(t) = a(\mathbb{1}_{[0,t]}\langle e_k|), \quad a_l^k(t) = a^\circ(\mathbb{1}_{[0,t]|e_l\rangle\langle e_k|}),$$

  where (e_k) is an orthonormal basis of $\mathbb{C}^d$.
- The Hudson-Parthasarathy quantum stochastic calculus allows to integrate with respect to the **quantum noises**:

$$da_k^0(t) = a^*(\mathbb{1}_{[t,t+dt]}|e_k\rangle), \quad da_0^k(t) = a(\mathbb{1}_{[t,t+dt]}\langle e_k|), \quad da_l^k(t) = a^\circ(\mathbb{1}_{[t,t+dt]|e_l\rangle\langle e_k|}).$$

- $M_{B_t^k}$ and $M_{N_t^k}$ are given explicitly in terms of the quantum noises as:

$$M_{B_t} = a_k^0(t) + a_0^k(t),$$

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- Consequently $M_{dB_t^k}$ reads:

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The Hudson-Parthasarathy Equation in the general situation

Main idea: The **quantum noises** behave nicely, in a way which is closed to classical noises. The quantum stochastic calculus is built with respect to those noises in a natural way.

Theorem (Hudson-Parthasarathy, 1984)

Write $\Lambda = \{1, ..., d\}$. Let $H, L_k, S_l^k \in \mathcal{B}(\mathcal{H})$ be such that $H = H^*$ and $\mathbb{S} = (S_l^k)_{k,l \in \Lambda}$ is a unitary operator on $\mathcal{H} \otimes \mathbb{C}^d$. Then the unitary Hudson-Parthasarathy Equation:

$$U_0 = I, \qquad dU_t = -\left(iH + \frac{1}{2}\sum_{k\in\Lambda} L_k^* L_k\right) U_t dt + \sum_{k\in\Lambda} L_k U_t da_k^0(t)$$

$$+ \sum_{k\in\Lambda}\left(-\sum_{l\in\Lambda} L_l^* S_l^k\right) U_t da_0^k(t) + \sum_{k,l\in\Lambda}\left(S_l^k - \delta_{k,l} I_{\mathcal{H}}\right) U_t da_l^k(t)$$

has a unique unitary solution on $\mathcal{H} \otimes \Phi(\mathbb{C}^d)$.

We need a way to characterize when the noise in the previous equation is in fact a **classical noise**.

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The Noise Algebra

Definition

*For all $t > 0$, the **Noise Algebra** $\mathcal{A}_t(U)$ is defined as the **smallest von Neumann algebra** on $\Phi(\mathbb{C}^d)$ such that*

$$U_s \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_t(U) \qquad \forall 0 \leq s \leq t$$

- It is a well-known result on von Neumann algebra that a **commutative von Neumann algebra** is always isomorphic to a commutative algebra $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- In this case, $\mathcal{A}_t(U)$ commutative for all $t > 0$ means that there exist a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ on it such that:

$$\mathcal{A}_t(U) \approx L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$$

- The problem is thus:

1) to associated the probability space with a stochastic process $(X_t)_{t\geq 0}$ on it (adapted to the filtration);
1) to identify this process;
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The case of a commutative environment

Theorem

Suppose that $\mathcal{A}_t(U_\cdot)$ is commutative. Then $\mathbb{C}^d = \mathcal{K}_{\mathbb{W}} \oplus \mathcal{K}_{\mathbb{P}}$ and Λ can be split into two subsets $\Lambda_{\mathbb{W}}$ and $\Lambda_{\mathbb{P}}$ such that the HP Equation takes the form:

$$dU_t = -iHU_t dt + dU_t^{\mathbb{W}} + dU_t^{\mathbb{P}},$$

where $U_\cdot^{\mathbb{W}}$ and $U_\cdot^{\mathbb{P}}$ are respectively the solutions of the HP Equations:

$$dU_t^{\mathbb{W}} = \sum_{k\in\Lambda_{\mathbb{W}}} \left(-\frac{1}{2}L_k^2 U_t^{\mathbb{W}} dt + L_k U_t^{\mathbb{W}} dB_t^k\right), \text{ on } \mathcal{H}\otimes\Phi(\mathcal{K}_{\mathbb{W}})$$

$$dU_t^{\mathbb{P}} = \sum_{k\in\Lambda_{\mathbb{P}}} \left(-\frac{1}{2}\rho_k^2 \left(2I_{\mathcal{H}} - S_k - S_k^*\right) U_t^{\mathbb{P}} dt + \rho_k \left(S_k - I_{\mathcal{H}}\right) U_t^{\mathbb{P}} dN_t^k\right), \text{ on } \mathcal{H}\otimes\Phi(\mathcal{K}_{\mathbb{P}})$$

where the L_k and the S_k are respectively selfadjoint and unitary operators on $\mathcal{H}$, the ρ_k are positive real numbers and

1. *$(B_t)_{t\geq 0} = (B_t^1, \cdots, B_t^m)$ is a real m-dimensional Brownian motion,*
2. *$(N_t)_{t\geq 0} = (N_t^1, ..., N_t^d)$ is a $d-m$-dimensional Poisson process, each coordinate N_t^k being of intensity ρ_k.*
3. *$(B_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are two independent processes.*

Some notions about the structure of the Fock space

- In the HP-Equation:

$$dU_t = -\left(iH + \frac{1}{2}\sum_{k\in\Lambda} L_k^* L_k\right) U_t dt + \sum_{k\in\Lambda} L_k U_t da_k^0(t)$$
$$+ \sum_{k\in\Lambda}\left(-\sum_{l\in\Lambda} L_l^* S_l^k\right) U_t da_0^k(t) + \sum_{k,l\in\Lambda}\left(S_l^k - \delta_{k,l} I_{\mathcal{H}}\right) U_t da_l^k(t)$$

a special role is played by the **unitary operator on** $\mathcal{H}\otimes\mathbb{C}^d \approx \mathcal{H}^{\oplus d}$:

$$\mathbb{S} = (S_l^k)_{k,l\in\Lambda}.$$

- If $\mathbb{C}^d = \mathcal{K}_1 \oplus \mathcal{K}_2$, then the probabilist Fock space can be factorized as:

$$\Phi(\mathbb{C}^d) = \Phi(\mathcal{K}_1)\otimes\Phi(\mathcal{K}_2)$$

- If $\mathcal{H}\otimes\mathcal{K}_1$ **is stable by** $\mathbb{S}$, then the HP Equation takes the form:

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Decomposition between a classical and a quantum part

Definition

Let $\mathcal{K}_1$ be a subspace of $\mathbb{C}^d$ and write $\mathcal{K}_2 = \mathcal{K}_1^{\perp}$. We say that $\Phi(\mathcal{K}_1)$ is a Commutative Subsystem of the Environment if $\mathcal{K}_1 \neq \{0\}$ and:

- *both $\mathcal{H} \otimes \mathcal{K}_1$ and $\mathcal{H} \otimes \mathcal{K}_2$ are stable by $\mathbb{S}$. Consequently*

$$dU_t = -iHU_t dt + dU_t^1 + dU_t^2.$$

- *$\mathcal{A}_t(U^1)$ is commutative.*

Theorem (Decomposition Theorem)

There exists a decomposition $\mathbb{C}^d = \mathcal{K}_c \oplus \mathcal{K}_q$, with $\mathcal{K}_c$ and $\mathcal{K}_q$ stable by $\mathbb{S}$ so that:

$$dU_t = -iHU_t dt + dU_t^c + dU_t^q$$

Furthermore U^q does not have any Commutative Subsystem and, either $\mathcal{K}_c = \{0\}$, or:

- *$\Phi(\mathcal{K}_c)$ is a Commutative Subsystem of the Environment.*
- *If $\mathcal{K}$ is a subspace of $\mathbb{C}^d$ and $\Phi(\mathcal{K})$ is a Commutative Subsystem of the Environment then $\mathcal{K}$ is a subspace of $\mathcal{K}_c$.*

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Thank you for your attention