Effective dynamics for multiple coupled Bose-Einstein condensates

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Joint work with Alessandro Michelangeli

Many-body system of two species of identical bosons

Hilbert space

$$\mathcal{H}_{N_1,N_2} = \underbrace{L^2_{sym}(\mathbb{R}^{3N_1}, dx_1 \dots dx_{N_1})}_{=:A \text{ sector}} \otimes \underbrace{L^2_{sym}(\mathbb{R}^{3N_2}, dy_1 \dots dy_{N_2})}_{=:B \text{ sector}}$$

(Everything generalizes to the n-component case)

Hamiltonian

$$\begin{aligned} H_{N_1,N_2} = & \sum_{i=1}^{N_1} (h_1)_i + \sum_{i<j}^{N_1} V_1(x_i - x_j) && \text{particles of type } A \\ & + \sum_{r=1}^{N_2} (h_2)_r + \sum_{r<s}^{N_2} V_2(y_r - y_s) && \text{particles of type } B \\ & + \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} V_{12}(x_i - y_r) && \text{coupling term} \end{aligned}$$

E.g. $h_1, h_2 = (i\nabla + A)^2 + U_{trap}$, or $\sqrt{m^2 + (i\nabla + A)^2} + U_{trap}$

Consider the mean-field limit for large N_1, N_2 with $N_1/N_2 = const$.

Which mean-field scaling?

$$\frac{1}{N_1} \quad \frac{1}{N_2} \quad \frac{1}{N_1+N_2} \quad \frac{1}{(N_1 N_2)^{1/2}} \quad \ldots \quad (\Rightarrow \text{ kinetic} \sim \text{potential} \sim O(N_1+N_2))$$

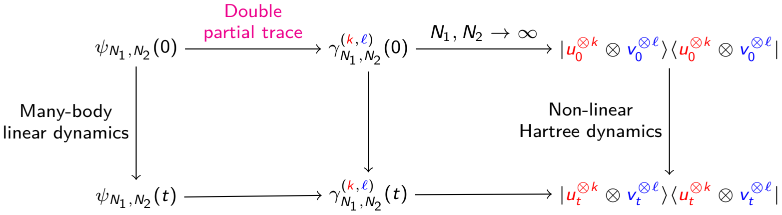
Physical heuristics select the choice

$$\begin{aligned} H_{N_1,N_2} = & \sum_{i=1}^{N_1} (h_1)_i^A + \frac{1}{N_1} \sum_{i<j}^{N_1} V_1(x_i - x_j) \\ & + \sum_{r=1}^{N_2} (h_2)_r^B + \frac{1}{N_2} \sum_{r<s}^{N_2} V_2(y_r - y_s) \\ & + \frac{1}{N_1+N_2} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} V_{12}(x_i - y_r) \end{aligned}$$

Partial traces and non-linear dynamics

Double partial trace $\psi_{N_1,N_2} \mapsto \gamma^{(k,\ell)}_{N_1,N_2} := \mathrm{Tr}_{k+1\ldots N_1,\ell+1\ldots N_2} |\psi_{N_1,N_2}\rangle\langle\psi_{N_1,N_2}|$

as a trace class operator on $L^2_{sym}(\mathbb{R}^{3k}, dx_1 \ldots dx_k) \otimes L^2_{sym}(\mathbb{R}^{3\ell}, dy_1 \ldots dy_\ell)$



Coupled non-linear Hartree system

$$\begin{cases} i\partial_t u_t = h_1 u_t + (V_1 * |u_t|^2)u_t + c_2(V_{12} * |v_t|^2)u_t \\ i\partial_t v_t = h_2 v_t + (V_2 * |v_t|^2)v_t + c_1(V_{12} * |u_t|^2)v_t \end{cases} \qquad c_i := \frac{N_i}{N_1 + N_2}$$

True in the one-component case for a wide class of potentials (Erdős-Yau, Bardos-Golse-Mauser, Erdős-Schlein, Knowles-Pickl).

Theorem (Michelangeli,O. 2016)

Suppose that

- $V_\alpha \in L^{p_\alpha} + L^{q_\alpha}$ *for some* $2 \leq p_\alpha \leq q_\alpha \leq \infty$, $\alpha = 1, 2, 12$;
- h_1, h_2, H_{N_1,N_2} *are self-adjoint and semibounded below;*
- *the Hartree system is energy well-posed (globally in time) with solutions* (u_t, v_t).

Then

$$\psi_{N_1,N_2}(0) = u_0^{\otimes N_1} \otimes v_0^{\otimes N_2}$$

implies

$$\mathrm{Tr}\left|\gamma^{(1,1)}_{N_1,N_2}(t) - |u_t \otimes v_t\rangle\langle u_t \otimes v_t|\right| \leq \frac{C(t)}{\sqrt{N_1 + N_2}}$$

Note: for the technique used here the convergence rate $(N_1 + N_2)^{-1/2}$ is optimal.

Based upon an adaptation of Pickl's counting method. Define the quantity

$$\alpha^{(1,1)}(t) := 1 - \left\langle u_t \otimes v_t \,,\, \gamma^{(1,1)}_{N_1,N_2}(t)\, u_t \otimes v_t \right\rangle$$

Look for an estimate

$$\partial_t \alpha^{(1,1)}(t) \lesssim \alpha^{(1,1)}(t) + \boxed{\frac{1}{N_1 + N_2}} \xRightarrow{\text{Grönwall}} \alpha^{(1,1)}(t) \lesssim \alpha^{(1,1)}(0) + \frac{1}{N_1 + N_2}$$

The result then follows since

$$\alpha^{(1,1)}_{N_1,N_2} \leq \mathrm{Tr}\left| \gamma^{(1,1)}_{N_1,N_2} - |u \otimes v\rangle\langle u \otimes v| \right| \lesssim \sqrt{\alpha^{(1,1)}_{N_1,N_2}}$$

With a simple computation

$$\partial_t \alpha^{(1,1)} = \mathrm{i}\, \langle \psi, \Big[H_{N_1,N_2} - \sum_{i=1}^{N_1} (h_1^u)_i - \sum_{r=1}^{N_2} (h_2^v)_r , 1 - |u \otimes v\rangle\langle u \otimes v| \Big] \psi \rangle,$$

where

$$h_1^u = h_1 + V_1 * |u|^2 + V_{12} * |v|^2 \qquad h_2^v = h_2 + V_2 * |v|^2 + V_{12} * |u|^2$$

$$\Downarrow$$

Cancellation of all single-particle/kinetic terms.
Therefore, allows for great generality of one-body (self-adjoint) hamiltonians

$$\Downarrow$$

Then estimate all terms containing potentials

Further remarks

- A complementary result [De Oliveira,Michelangeli 2016]
  - more restrictive on h_1, h_2, V_α and with weaker rate
  - focus also fluctuations around Hartree dynamics (Fock space method)
- Technique expected to reproduce also more singular/realistic scalings (GP regime)
- More difficult problem: possibility of transitions between the two particle populations