Approximation of Schrödinger operators with singular interactions supported on hypersurfaces

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Outline

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For a zero-set $\Sigma \subset \mathbb{R}^d$ and $\alpha : \Sigma \to \mathbb{R}$ consider

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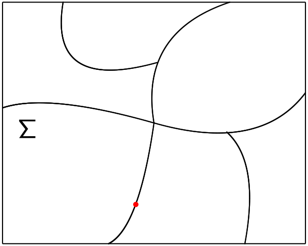

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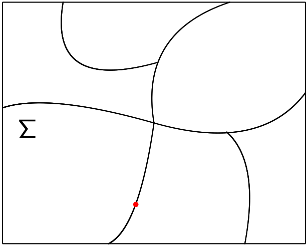

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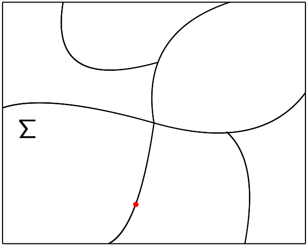

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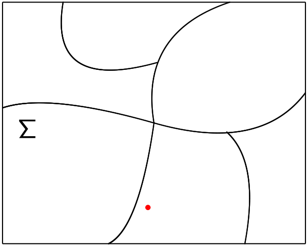

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  - δ-potentials describe interactions between the particles

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Some names: Albeverio, Behrndt, Brasche, Cacciapuoti, Carlone, Corregi, Dell’Antonio, Exner, Figari, Figotin, Finco, Gesztesy, Griesemer, Holden, Kondej, Kostenko, Kuchment, Kühn, M. Langer, Lotoreichik, Manko, Malamud, Michelangeli, Neidhardt, Nizhnik, Noja, Ourmières-Bonafos, Pankrashkin, Posilicano, Shkalikov, Teta, . . .

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- Show that $-\Delta - V_\varepsilon$ converge to $A_{\delta,\alpha}$ in suitable sense
- Then, spectral data of $A_{\delta,\alpha}$ and $-\Delta - V_\varepsilon$ are approximately the same

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2. δ-operators and their approximation

Schrödinger operators with δ-interactions on hypersurfaces

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- $\mathfrak{a}_{\delta,\alpha}$ is closed and bounded from below [Brasche, Exner, Kuperin, Šeba 94]
- Representing operator = $A_{\delta,\alpha}$
- It holds for $f \in \operatorname{dom} A_{\delta,\alpha}$ [Behrndt, Exner, M. Langer, Lotoreichik 13]:

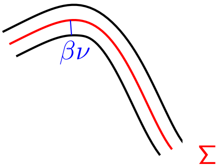
$$A_{\delta,\alpha} f = -\Delta f \quad \text{on} \quad \mathbb{R}^d \setminus \Sigma$$
$$\alpha f|_\Sigma = \big[\partial_\nu f|_\Sigma\big]$$

Construction of the approximating sequence

- Assume $\exists \beta > 0$ such that

  $$\Sigma \times (-\beta, \beta) \ni (x_\Sigma, t) \mapsto x_\Sigma + t\nu(x_\Sigma) \in \mathbb{R}^d$$

  is injective



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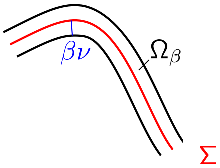

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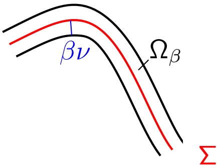

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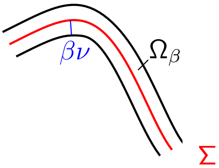

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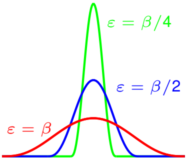

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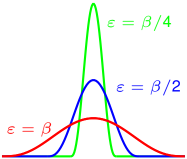

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- $-\Delta - V_\varepsilon$ is self-adjoint on $H^2(\mathbb{R}^d)$

Main result

Theorem ([Behrndt, Exner, H., Lotoreichik])

Define $\alpha \in L^\infty(\Sigma)$ *as*

$$\alpha(x_\Sigma) := \int_{-\beta}^{\beta} V(x_\Sigma + s\nu(x_\Sigma))\,\mathrm{d}s$$

f.a.a. $x_\Sigma \in \Sigma$ *and let* $\lambda \ll 0$. *Then there exists* $c > 0$ *such that*

$$\left\| (-\Delta - V_\varepsilon - \lambda)^{-1} - (A_{\delta,\alpha} - \lambda)^{-1} \right\| \leq c\varepsilon\big(1 + |\ln \varepsilon|\big)$$

for all sufficiently small $\varepsilon > 0$. *In particular* $-\Delta - V_\varepsilon$ *converge to* $A_{\delta,\alpha}$ *in the norm resolvent sense.*

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- $u(-\Delta - V_\varepsilon) \to u(A_{\delta,\alpha})$ strongly for any $u \in C_b(\mathbb{R})$

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Corollary

Let $Q \in L^\infty(\mathbb{R}^d)$ *be real-valued. Then*

$$\left\|(-\Delta - V_\varepsilon + Q - \lambda)^{-1} - (A_{\delta,\alpha} + Q - \lambda)^{-1}\right\| \to 0, \quad \varepsilon \to 0+$$

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- general space dimension $d \geq 2$
- general C^2-smooth Σ
- arbitrary interaction strength $\alpha \in L^\infty(\Sigma)$
- any potential $Q \in L^\infty(\mathbb{R}^d)$

Thank you for your attention!