Characteristic length of sequences via one-scale H-measures

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Joint work with Nenad Antonić and Martin Lazar

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Introduction

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If the defect measure is not trivial we need another objects to determine all the properties of the sequence:

- H-measures
- semiclassical measures
- ...

$\Omega \subseteq \mathbf{R}^d$ open.

Theorem

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_H \in \mathcal{M}(\Omega \times \mathrm{S}^{d-1}; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\Big) \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_H, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle\,.$$

(Unbounded) Radon measure $\boldsymbol{\mu}_H$ *we call the H-measure corresponding to the (sub)sequence* (u_n).

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The Wigner transform

(u_n) from $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, $\omega_n \to 0^+$,

$$\mathbf{W}_n(\mathbf{x},\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\xi}}\, \mathsf{u}_n\left(\mathbf{x}+\frac{\omega_n \mathbf{y}}{2}\right) \otimes \mathsf{u}_n\left(\mathbf{x}-\frac{\omega_n \mathbf{y}}{2}\right) d\mathbf{y}$$

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$$\mathbf{W}_{n'} \xrightarrow{\mathcal{S}'} \boldsymbol{\mu}_{sc}^{(\omega_{n'})}\,.$$

[G2] Patrick Gérard: *Mesures semi-classiques et ondes de Bloch, Sem. EDP 1990–91 (exp. 16)*, (1991)

[LP] Pierre Louis Lions, Thierry Paul: *Sur les measures de Wigner, Revista Mat. Iberoamericana* **9,** (1993) 553-618

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$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$,

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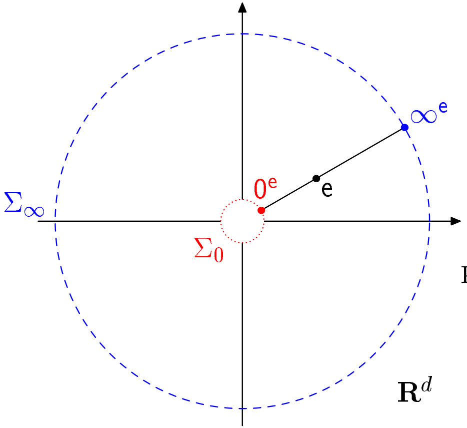

$$\Sigma_0 := \{0^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$
$$\Sigma_\infty := \{\infty^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$
$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary

a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

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Theorem

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d);\mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1,\varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big((\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi})\Big)\psi(\omega_{n'}\boldsymbol{\xi})\,d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \Big\rangle\,.$$

(Unbounded) Radon measure $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ *we call the one-scale H-measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* (u_n).

[T2] Luc Tartar: *The general theory of homogenization: A personalized introduction*, *Springer* (2009)

[T3] Luc Tartar: *Multi-scale H-measures*, *Discrete and Continuous Dynamical Systems, S* **8** (2015) 77–90.

[AEL] Nenad Antonić, M.E., Martin Lazar: *Localisation principle for one-scale H-measures*, submitted (`arXiv:1504.03956`).

Theorem

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

c) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

d) $\mathrm{tr}\,\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff$ (u_n) *is* (ω_n) *– oscillatory*

Theorem

$\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi \rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi \rangle$,

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi \circ \boldsymbol{\pi} \rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi \rangle$,

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Example 1 revisited

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_0 & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Localisation principle - assumptions

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $l \in 0..m$
- $\varepsilon_n > 0$ bounded
- $\mathbf{A}_n^{\boldsymbol{\alpha}} \to \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$$

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

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- $l \in 0..m$
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For $l = 0$ the condition on (f_n) is equivalent to

$$(\forall \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-m}_{\varepsilon_n}} \to 0 \,,$$

where $\|\mathsf{u}\|^2_{\mathrm{H}^s_h} = \int_{\mathbf{R}^d} (1 + 2\pi |h\boldsymbol{\xi}|^2)^s |\hat{\mathsf{u}}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}$ is the semiclassical norm of $\mathsf{u} \in \mathrm{H}^s(\Omega; \mathbf{R}^d)$.

Theorem

For $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$, *corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}_c(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}_\infty(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Example: equations with characteristic length (1/2)

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$\begin{cases} u_n^1 + \varepsilon_n \partial_{x_1}(a_1 u_n^1) = f_n^1 \\ u_n^2 + \varepsilon_n \partial_{x_2}(a_2 u_n^2) = f_n^2 \end{cases},$$

where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-1}_{\varepsilon_n}} \to 0\,,$$

while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

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where $\varepsilon_n \to 0^+$, $\mathsf{f}_n := (f_n^1,\ f_n^2) \in \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \|\varphi \mathsf{f}_n\|_{\mathrm{H}^{-1}_{\varepsilon_n}} \to 0\,,$$

while $a_1, a_2 \in \mathrm{C}(\Omega; \mathbf{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) (i.e. $c = 1$) associated to (u_n) we get the relation

$$\left(\frac{1}{1+|\boldsymbol{\xi}|}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{2\pi i \xi_1}{1+|\boldsymbol{\xi}|}\begin{bmatrix}a_1(\mathbf{x}) & 0\\ 0 & 0\end{bmatrix} + \frac{2\pi i \xi_2}{1+|\boldsymbol{\xi}|}\begin{bmatrix}0 & 0\\ 0 & a_2(\mathbf{x})\end{bmatrix}\right)\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i \frac{2\pi \xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x}) \right) \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,,$$

hence

$$\frac{1}{1+|\boldsymbol{\xi}|} \mu^{11}_{\mathrm{K}_{0,\infty}} = 0\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|} \mu^{11}_{\mathrm{K}_{0,\infty}} = 0$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^2)$ satisfies

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hence

$$\operatorname{supp}\mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \Sigma_\infty\,, \qquad \frac{\xi_1}{1+|\boldsymbol{\xi}|}\mu^{11}_{\mathrm{K}_{0,\infty}} = 0$$

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hence

$$\operatorname{supp} \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \Sigma_\infty\,, \quad \operatorname{supp} \mu^{11}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})$$

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whose $(1,1)$ component reads

$$\left(\frac{1}{1+|\boldsymbol{\xi}|} + i\frac{2\pi\xi_1}{1+|\boldsymbol{\xi}|} a_1(\mathbf{x})\right) \mu_{\mathrm{K}_{0,\infty}}^{11} = 0\,,$$

hence

$$\operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times \Sigma_\infty\,, \quad \operatorname{supp} \mu_{\mathrm{K}_{0,\infty}}^{11} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})$$

Let $\Omega \subseteq \mathbf{R}^2$ be open, and let $\mathsf{u}_n := (u_n^1, u_n^2) \longrightarrow 0$ in $\mathrm{L}^2_{\rm loc}(\Omega; \mathbf{C}^2)$ satisfies

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Analogously, from the $(2,2)$ component we get

$$\operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} \subseteq \Omega \times \{\infty^{(-1,0)}, \infty^{(1,0)}\},$$

hence $\operatorname{supp}\mu^{11}_{\mathrm{K}_{0,\infty}} \cap \operatorname{supp}\mu^{22}_{\mathrm{K}_{0,\infty}} = \emptyset$ which implies $\mu^{12}_{\mathrm{K}_{0,\infty}} = \mu^{21}_{\mathrm{K}_{0,\infty}} = 0$.

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The very definition of one-scale H-measures gives $u^1_n \bar{u}^2_n \xrightharpoonup{*} 0$.

This approach can be systematically generalised by introducing a variant of compensated compactness suitable for problems with characteristic length.

Let $\mathsf{u}_n \longrightarrow \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfy

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ in $\mathrm{C}(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$, let $\varepsilon_n \to 0^+$, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^q)$ be such that for any $\varphi \in \mathrm{C}^\infty_c(\Omega)$

$$\frac{\widehat{\varphi \mathsf{f}_n}}{1 + k_n}$$

is precompact in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^q)$. Furthermore, let $Q(\mathbf{x}; \boldsymbol{\lambda}) := \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$, where $\mathbf{Q} \in \mathrm{C}(\Omega; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$, $\mathbf{Q}^* = \mathbf{Q}$, is such that $Q(\cdot; \mathsf{u}_n) \xrightarrow{*} \nu$ in $\mathcal{M}(\Omega)$.
Then we have

a) $(\exists\, c \in [0, \infty])(\forall\, (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))(\forall\, \boldsymbol{\lambda} \in \Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\ Q(\mathbf{x}; \boldsymbol{\lambda}) \geqslant 0 \implies \nu \geqslant Q(\cdot, \mathsf{u})$,

b) $(\exists\, c \in [0, \infty])(\forall\, (\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))(\forall\, \boldsymbol{\lambda} \in \Lambda_{c;\mathbf{x},\boldsymbol{\xi}})\ Q(\mathbf{x}; \boldsymbol{\lambda}) = 0 \implies \nu = Q(\cdot, \mathsf{u})$,

where

$$\Lambda_{c;\mathbf{x},\boldsymbol{\xi}} := \{\boldsymbol{\lambda} \in \mathbf{C}^r : \mathbf{p}_c(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\lambda} = 0\} \,,$$

and $\mathbf{p}_c$ is given as before.