Multi-vortex Solutions of the Ginzburg-Landau

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GL Equations

The Ginzburg-Landau (GL) equations of superconductivity are

$$-\Delta_A \psi + \kappa^2(|\psi|^2 - 1)\psi = 0$$
$$\mathrm{curl}^* \,\mathrm{curl}\, A - \mathrm{Im}(\bar{\psi}\nabla_A \psi) = 0$$

where

$$\psi : \mathbb{R}^2 \to \mathbb{C}$$
$$A : \mathbb{R}^2 \to \mathbb{R}^2$$

and κ is the GL parameter.

Abrikosov Lattice States

Abrikosov lattice states are those whose physical quantities are periodic with respect to some lattice. Key quantities includes

1. The magnetic field $B = \operatorname{curl} A$
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We classify solutions by flux per unit cell.

Solutions

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We can bifurcate a mixed state from the normal state.

Theorem (I.M. Sigal, T. Tzaneteas (2011))

Under suitable assumptions on parameters, a unique lattice solution (upto symmetry), with 1 flux per unit cell, exists in a neighbourhood of the normal branch.

Remark: asymptotics, in terms of the bifurcation parameter, are also given.

Idea of the Proof: Bifurcation via Lyapunov-Schmidt Procedure

Fix a lattice $\mathcal{L} = \mathbb{Z} + \tau\mathbb{Z}$ for $\operatorname{Im}\tau > 0$ and a flux $2\pi n = \int_\Omega \operatorname{curl} A$. The linearized GL equations at a normal state $(0, A_b)$ such that $\operatorname{curl} A_b = b$ is

$$\begin{pmatrix} -\Delta_{A_b} - \kappa^2 & 0 \\ 0 & \operatorname{curl}^* \operatorname{curl} \end{pmatrix} \begin{pmatrix} \psi' \\ A' \end{pmatrix} = 0$$

Linear Analysis

The operator curl* curl has an infinite dimensional kernel, namely the ones of the form $\nabla\chi$. By appropriate gauge fixing, we can consider A of the form $a_b + \alpha := \frac{b}{2}J + \alpha$, where

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One can further show that $\mathrm{curl}^* \mathrm{curl}$ on such space is positive definite.

Linear Analysis

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$$\theta(z) = e^{\frac{b}{4}(|z|^2 - z^2)}\psi(z)$$

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is holomorphic. When periodic condition is taken into consideration, the ground state space becomes finite dimensinoal, whose dimension equals n. This associated space of theta functions will be denoted by V_n.

Remark: periodicity condition translates to the theta functions as

$$\theta(z+1) = \theta(z)$$
$$\theta(z+\tau) = e^{-2\pi i n z} e^{-\pi i n \tau} \theta(z)$$

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Methods:

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Methods:

1. Topological degree theory. Global gauge freedom essentially gives $2n - 1$ real dimensional kernel for the linear operator.
2. Use additional symmetries to reduce the dimension to of V_n to 1.

The Immediate Problem

The simplest symmetry would to require the physical quantities be periodic with respect to a finer lattice. But this gives no new results as the solution is in some sense reducible.

A Solution

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Observation 2: Not all rotations are allowed. Only those that leaves the lattice invariant are possible candidates. $\Rightarrow$ So we consider hexagonal lattice for allowing maximal rotation group.

We want to solve GL equations for ψ and α over these two spaces:

$$L^2_{\xi,\eta} = \{\psi \in L^2(\Omega_{WS}; \mathbb{C}) : \psi(\xi x) = \eta\psi(x)\}$$
$$\vec{L}^2_{\xi} = \{\alpha \in L^2(\Omega_{WS}, \mathbb{R}^2) : \langle\alpha\rangle = 0,\ \text{div}\alpha = 0,\ \alpha(R_\xi x) = R_\xi\alpha(x)\}$$

for some $\eta \in S^1 \subset \mathbb{C}$ and $\xi = e^{2\pi i/6}$.

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Now, consider the linear problem again. Translating the condition on ψ to theta functions. We see

$$\theta(\xi z) = \eta e^{-i\pi n\xi z^2}\theta(z)$$

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Remark: it is not clear any such θ should exist at first sight.
Define for each $\theta : \mathbb{C} \to \mathbb{C}$,

$$T_{\xi,n}(\theta)(z) := e^{i\pi n\xi z^2}\theta(\xi z)$$

Lemma

$T_{\xi,n}$ is a linear endomorphism on V_n if n is even.

Since $T_{\xi,n}^6 = 1$, it can be completely diagonalized.

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Moreover, expanding the relation

$$T_{\xi,n}(\theta)(z) = e^{i\pi n\xi z^2}\theta(\xi z) = \lambda\theta(z)$$

in z at the origin, we see that $\lambda = \xi^k$, where k is the multiplicity of zeros of θ at the origin.

Theorem

Every C_6-equivariant theta function is a product of the following theta functions (upto scaling):

1. *This theta function has a single zero of multiplicty 2 at the orgin* – **We define this to be θ_2 from now on.**
2. *This theta function has 2 simple zeros located on the left most 2 vertices of the Wigner-Seitz cell* – **We define this to be θ_0 from now on.**
3. *This theta function has 4 simple zeros. One zero is at the origin, the other three are on the midpoint of the left most three edges of the Wigner-Seitz cell* – **We define this to be θ_1 from now on.**
4. *This theta function has 6 simple zeros on $W - W^o$, forming an orbit of C_6.*
5. *This theta function has 6 simple zeros on W^o, forming an orbit of C_6*

Using this theorem, we see that

Lemma

With the definition of $\theta_0, \theta_1, \theta_2$ as above. We have that

$$T_{\xi,2}\theta_0 = \xi^0\theta_0$$
$$T_{\xi,4}\theta_1 = \xi^1\theta_1$$
$$T_{\xi,2}\theta_2 = \xi^2\theta_2$$

Vortex Number	Eigenvalue	Eigenvectors
$n = 2$	1	θ_0
	ξ^2	θ_2
$n = 4$	1	θ_0^2
	ξ	θ_1
	ξ^2	$\theta_0\theta_1$
	ξ^4	θ_2^2
$n = 6$	1	$\theta_0^3, \ \theta_2^3$
	ξ	$\theta_0\theta_1$
	ξ^2	$\theta_0^2\theta_2$
	ξ^3	$\theta_1\theta_2$
	ξ^4	$\theta_0\theta_2^2$

Vortex Number	Eigenvalue	Eigenvectors
$n = 8$	1	$\theta_0^4,\ \theta_0\theta_2^3$
	ξ	$\theta_0^2\theta_1$
	ξ^2	$\theta_2^4,\ \theta_1^2,\ \theta_0^3\theta_2$
	ξ^3	$\theta_0\theta_1\theta_2$
	ξ^4	$\theta_0^2\theta_2^2$
	ξ^5	$\theta_1\theta_2^2$
$n = 10$	1	$\theta_0^5,\ \theta_0^2\theta_2^3$
	ξ	$\theta_0^3\theta_1,\ \theta_1\theta_2^3$
	ξ^2	$\theta_0^4\theta_2,\ \theta_0\theta_2^4,\ \theta_0\theta_1^2$
	ξ^3	$\theta_0^2\theta_1\theta_2$
	ξ^4	$\theta_0^3\theta_2^2,\ \theta_2^5,\ \theta_1^2\theta_2$
	ξ^5	$\theta_0\theta_1\theta_2^2$

Theorem

Under suitable conditions on parameters, a solution with 2,4,6,8, or 10 vortices per unit cell exists near the normal state. Moreover, they cannot be regarded as solutions to a finer lattice.

Future Outlook

One can try to solve the equation on

$$L^2_{\xi,\eta} = \{\psi \in L^2(\Omega_{WS};\mathbb{C}) : \psi(\xi x) = \eta\psi(x)\}$$
$$\vec{L}^2_{\xi} = \{\alpha \in L^2(\Omega_{WS},\mathbb{R}^2) : \langle\alpha\rangle = 0,\ \mathrm{div}\alpha = 0,\ \alpha(R_\xi x) = R_\xi\alpha(x)\}$$

for different ξ, η. In fact

Theorem ($\xi = \eta = -1$)

Suppose that $n = p$ is an odd prime. Then, the space of odd theta function is $\frac{p-1}{2}$. Moreover, no odd theta function in V_p is quasi-periodic with respect to a finer lattice of the form

$$L_{a,b} := \frac{1}{a}\mathbb{Z} + \frac{\tau}{b}\mathbb{Z}$$

where $a, b \in \mathbb{Z}$.

Future Outlook

1. Any other ξ, η which works?
2. Rotational symmetry about a different point on the lattice not the origin?
3. Square lattice?
4. Result for general lattice?

Thank you for your attention!