Chernoff equivalence
for the methods of averaging of semigroups, generating by the Schrödinger-type operators

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Structure

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Motivaton

In general concept of a linear quantization the kernel $\tilde{H}(x, y)$ of operator $\hat{H}$ corresponding to its classical symbol $H(q, p)$ is given by the formula:

$$\tilde{H}(x, y) = \int_{\Gamma} K(q, p|x, y) H(q, p) dq dp,$$

where Γ is a phase space of classical system, $q, p \in R^n$ are phase coordinates, $\hat{H} \in \mathcal{H}(L_2(R^n))$, and quantization kernel is given by the formula:

$$K(q, p|x, y) = \frac{1}{2\pi} \int_0^1 \delta(q - \tau x - (1 - \tau) y) e^{ip(x-y)} Q(\tau) d\tau,$$

Motivaton

L. A. Borisov, Yu. N. Orlov Analyzing the dependence of finite-fold approximations of the harmonic oscillator equilibrium density matrix and of the Wigner function on the quantization prescription. TMPh, 184:1 ('15)

List of the quantization rules:

- Weyl quantization: $Q(\tau) = \delta(\tau - 1/2)$
- Jordan quantization: $Q(\tau) = \frac{1}{2}(\delta(\tau) + \delta(\tau - 1))$
- Born quantization: $Q(\tau) = 1$

In general:

$$\tilde{H}(x, y) = \int_0^1 \tilde{H}_\tau(x, y) Q(\tau) d\tau,$$

How to costruct formulas for $\tilde{H}$ in terms of $\tilde{H}_\tau$,

where $\tilde{H}$ is obviously an average value of $\tilde{H}_\tau$?

Chernoff theorem

Chernoff P.R., Note on product formulas for operator semigroups, JFA. ('68)

Theorem

Let X be a Banach space. Let $F : [0,\infty) \to L(X)$ be a strongly continuous mapping such that $F(0) = I$, $\|F(t)\| \leq \exp(at)$ for some $a \in R$, D be a linear subspace in $D(F'(0))$ and the restriction of $F'(0) \to D$ be a closable operator whose closure we denote by C. If C is the generator of a stronly continuous semigroup $\exp(tC)$, then $F(t/n)^n$ converges to $\exp(tC)$ as $n \to \infty$ in the strong operator topology uniformly with respect to $t \in [0, T]$ for each $T > 0$.

STT Formula

O.G.Smolyanov, A.G.Tokarev, A.Truman Hamiltonian Feynman path integrals via the Chernoff formula, J.Math.Phys., **43**:10 ('02)

$$\exp(t\hat{H}) = \lim_{n\to\infty} (\widehat{\exp(\frac{tH}{n})})^n,$$

where H is a classical symbol of $\hat{H}$

Chernoff equivalence

Yu. N. Orlov, V. Zh. Sakbaev, O. G. Smolyanov Feynman formulas as a method of averaging random Hamiltonians, Proceedings of the Steklov Institute of Mathematics. **285** ('14)

Definition

The operator-valued functions **F**, **G** $\in \Pi$ is said to be Chernoff equivalent if for every $T > 0$ and every $u \in X$ the condition $\lim_{n\to\infty} \sup_{t\in[0,T]} \|((\mathbf{G}(\frac{t}{n}))^n - (\mathbf{F}(\frac{t}{n}))^n)u\| = 0$ is satisfied.

Π is a set of strongly continuous operator-valued functions $F : [0, \infty) \to B(X)$ that satisfy the condition $F(0) = \mathbf{I}$ and the condition $\|F(t)\|_{B(x)} \leq 1 + ct, t \in [0, \delta)$, for some $c \geq 0$ and have a derevative $F'(0)$ at zero whose closure serves as a generator of a strongly continuous semigroup.

Problem Statement

Let a partial Hamiltonian $\hat{H}_\tau$ generates a one parameter semigroup and average Hamiltonian $\hat{H}$ generates a semigroup for corresponding Cauchy problem for Schrödinger equation.

In what sense can this semigroup be treated as an average semigroup?

Averaging Theorem

Theorem

Let A_n be a sequence of self-adjoint operators in a Hilbert space H. Let μ_n be a sequence of nonnegative numbers such that the sum of the series of these numbers is equal to one. Suppose that there exists a number $m \in N$ such that for all $n \geq m+1$ the operators A_n are bounded and the series $\sum_{k=m+1}^{\infty} \mu_k \|A_k\|$ converges. Suppose also that there exists a linear subspace $D \subset H$ that is an essential domain of each of the operators $A_n, n \in 1, ...m$ and $S_n = \sum_{k=1}^{n} \mu_k A_k, n \in 1, ...m$ Then the mean value of the random semigroup $F(t) = \sum_{n \in N} \exp(-itA_n)\mu(n), t \geq 0$ is Chernoff equivalent to the unitary group $U(t) = exp(-itS), t \in R$, where $S = \sum_{k=1}^{\infty} \mu_k A_k$.

Proposition 1

Let $H = L_2(R)$ and for any $\varepsilon \in R$ and any $v \in R$ there exists a family (not a semigroup) of transformations $\mathbf{U}_{\varepsilon,v}(t),\ t \geq 0$ of a set H in according to the formula

$$\mathbf{U}_{\varepsilon,v}(t)u(x) = u(x + vt + \varepsilon t^{1/2}),\ t \geq 0.$$

Let also there exists on R a probabilistic measure μ with a density p_μ such that p_μ is even function and there exists a second and a third finite moments $(\int_R \varepsilon^2 p_\mu(\varepsilon)d\varepsilon = D > 0,\ \int_R |\varepsilon|^3 p_\mu(\varepsilon)d\varepsilon < \infty)$.

Then for any $v \in R$ a family of averaging transformations

$$\mathbf{U}_v^\mu(t) = \int_R \mathbf{U}_{\varepsilon,v}(t)p_\mu(\varepsilon)d\varepsilon,\ t \geq 0,$$

is Chernoff equivalent to a semigroup solving the Cauchy value problem for the Heat equation

$$u'_t = Du''_{xx} + vu'_x,\ \ t > 0,\ x \in R; \quad u|_{t=+0} = u_0.$$

Proposition 2

Let $H = L_2(R)$ and for any $\varepsilon \in R$ there exists a family (not a semigroup) of transformations $\mathbf{U}_\varepsilon(t),\ t \geq 0$ of a set H in according to the formula

$$\mathbf{U}_\varepsilon(t)u(x) = u(x + \varepsilon t^{1/2}),\ t \geq 0.$$

Let there exists a pseudomeasure μ on R with density

$$p_\mu(x) = \frac{e^{i\pi/4}}{\sqrt{4\pi D}} e^{\frac{i}{4D}x^2},\ x \in R$$

Then a family of averaging transformations

$$\mathbf{U}^\mu(t) = \int_R \mathbf{U}_\varepsilon(t) d\mu(\varepsilon),\ t \geq 0,$$

is Chernoff-equivalent to a semigroup (and even coincides with it) solving the Cauchy problem for the Schrödinger equation

$$iu'_t = Du''_{xx},\ t > 0,\ x \in R; \quad u|_{t=+0} = u_0.$$

Proposition 3

Let $H = L_2(R)$ and for any $\varepsilon \in R$ there exists a family of transformations $\mathbf{U}_\varepsilon(t),\ t \geq 0$ of a set H, in according to the formula

$$\mathbf{U}_{\varepsilon,\sigma}(t)u(x) = u(x + \varepsilon t^\sigma),\ t \geq 0;\ \sigma > 0.$$

Let there exists an alternating measure μ on R, such that Fourier transformation with respect to Lebesgue measure of its density p_μ is defined by the equation

$$\hat{p}_\mu(\xi) = e^{-D|\xi|^{2\alpha}}, \xi \in R, \alpha > 0$$

Then in case $\sigma = \frac{1}{2\alpha}$ a family of averaging transformations

$$\mathbf{U}^\mu_\sigma(t) = \int\limits_R \mathbf{U}_{\varepsilon,\sigma}(t) d\mu(\varepsilon),\ t \geq 0,$$

is Chernoff equivalent to a semigroup (and even coincides with it) solving the Cauchy problem for the fractional-order diffusion equation

$$u'_t = -D(-\triangle)^\alpha u,\ \ t > 0,\ x \in R; \quad u|_{t=+0} = u_0.$$

Proposition 4

Let $H = L_2(R)$ and for any $\varepsilon \in R$ there exists a family of transformations $\mathbf{U}_\varepsilon(t),\ t \geq 0$ of a set H in according to the formula

$$\mathbf{U}_\varepsilon(t)u(x) = u(x + \varepsilon t),\ t \geq 0.$$

Let there exists a one-parameter family of alternating measures $\mu(t),\ t \geq 0$ on R, such that Fourier transformation with respect to Lebesgue measure of its density $p_{\mu(t)}$ is defined by the equation

$$\hat{p}_{\mu(t)}(\xi) = e^{-D\sqrt{|\xi|^2 + t^2}},\ \xi \in R$$

Then a family of averaging transformations

$$\mathbf{U}^\mu(t) = \int_R \mathbf{U}_\varepsilon(t) d\mu(\varepsilon),\ t \geq 0,$$

is Chernoff-equivalent to a semigroup solving the Cauchy problem for the equation

$$u'_t = -D\sqrt{(-\triangle) + \mathbf{I}}u,\ t > 0,\ x \in R; \quad u|_{t=+0} = u_0.$$

Proposition 5

Let $H = L_2(R)$ and for any $\varepsilon \in R$ there exists a family of transformations $\mathbf{U}_\varepsilon(t),\ t \geq 0$ of a set H in according to the formula

$$\mathbf{U}_\varepsilon(t)u(x) = u(x + \varepsilon t),\ t \geq 0.$$

Let there exists a one-parameter family of complex pseudomeasures $\mu(t),\ t \geq 0$ on R such that Fourier transformation with respect to Lebesgue measure of its density $p_{\mu(t)}$ is defined by the equation

$$\hat{p}_{\mu(t)}(\xi) = e^{-iD\sqrt{|\xi|^2 + t^2}},\ \xi \in R$$

Then a family of averaging transformations

$$\mathbf{U}^\mu(t) = \int_R \mathbf{U}_\varepsilon(t) d\mu(\varepsilon),\ t \geq 0,$$

is Chernoff-equivalent to a semigroup solving the Cauchy problem for the equation

$$u'_t = -iD\sqrt{(-\triangle) + \mathbf{I}}u,\ \ t > 0,\ x \in R; \quad u|_{t=+0} = u_0.$$

Summary

- The averaging procedure of one-parameter semigroups, based on Chernoff equivalence for operator-functions is constructed.
- The initial value problem solutions are investigated for fractional diffusion equation and for Schrödinger equation with relativistic Hamiltonian of freedom motion.
- It is established that in these examples the quantization can be treated as averaging of random translation operators in classical coordinate space.

Grazie per l'attenzione!