Entanglement Rates in Bipartite Systems

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Outline of the talk

- Entangling Rate in Closed Systems
- Mixing Rate Problem
- Entangling Rate in Open Systems
  - Relative Entropy of Entanglement in ancilla-free system
  - Quantum Mutual Information in ancilla-assisted system
- Future Research

Entangling Rate in Closed System

	Alice		**Bob**
systems	A	$\leftarrow H_{AB} \rightarrow$	B
ancillas	a		b

initial state

$$\rho(0) = |\Psi\rangle\langle\Psi|_{aABb}$$

Time evolution in Schrödinger picture

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] = -i[\mathbb{1}_a \otimes H_{AB} \otimes \mathbb{1}_b, \rho(t)]$$

has an explicit solution

$$\rho(t) = U^*(t)\,|\Psi\rangle\langle\Psi|\,U(t),$$

where

$$U(t) = e^{itH} = I_a \otimes e^{itH_{AB}} \otimes I_b$$

is a unitary evolution. The state $\rho(t)$ is always pure.

To measure the entanglement between Alice and Bob, we calculate the **entanglement entropy**

$$E(\rho(t)) := S(\rho_{aA}(t)) = -\mathrm{Tr}\, \rho_{aA}(t) \ln \rho_{aA}(t),$$

here $\rho_{aA}(t) = \mathrm{Tr}_{Bb}\, \rho_{aABb}(t) = \mathrm{Tr}_{Bb}\, U^*(t)\, |\Psi\rangle \langle\Psi|\, U(t)$.

Remark

Small Total Entangling (Bennet et al) The total change of the entanglement $E(\rho(t))$ is at most $2 \ln d$, where $d = \min\{|A|, |B|\}$.

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The **entangling rate** is defined by

$$\Gamma(\Psi, H) = \left.\frac{dE(\rho(t))}{dt}\right|_{t=0}.$$

It can be expressed as

$$\Gamma(\Psi, H) = -i\, \mathrm{Tr}\Big(H_{AB}[\rho_{aAB}, \ln(\rho_{aA}) \otimes I_B]\Big).$$

Conjectured by Bravyi '07:

Theorem

Small Incremental Entangling
There is a universal constant c such that for all dimensions of ancillas a, b and for all states $|\Psi\rangle$, the following holds

$$\Gamma(\Psi, H) \leq c\|H\| \ln d,$$

where $d = \min\{|A|, |B|\}$.

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History of results:

- 2003 Bennett et al.: $\Gamma(\Psi, H) \leq c\|H\|d^4$
- 2007 Bravyi: $\Gamma(\Psi, H) \leq 2\|H\|d^2$
- 2007 Bravyi, no ancillas: $\Gamma(\Psi, H) \leq c(d)\|H\| \log d$, with $c(d) \to 1$ with large d
- 2013 Lieb, Vershynina: $\Gamma(\Psi, H) \leq (4/\ln 2)\|H\|d$
- 2013 Van Acoleyen, Mariën, Verstraete: $\Gamma(\Psi, H) \leq 18\|H\| \log d$
- 2013 Audenaert: $\Gamma(\Psi, H) \leq 8\|H\| \log d$

Mixing Rate Problem

$\mathcal{H}$ is a Hilbert space of dimension d. Let $\mathcal{E}_2 = \{(p, \rho_1), ((1-p), \rho_2)\}$ be a probabilistic ensemble on $\mathcal{H}$ with expected density operator

$$\rho = p\rho_1 + (1-p)\rho_2.$$

For any Hamiltonian H the time-dependent state is

$$\rho(t) = p\rho_1 + (1-p)e^{-itH}\rho_2 e^{itH}.$$

The von Neumann entropy of this state is

$$S(\rho(t)) = -\mathrm{Tr}\,\rho(t)\ln\rho(t).$$

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Small Total Mixing For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), ((1-p), \rho_2)\}$, the entropy of a state $\rho(t)$ at any time t satisfies

$$\overline{S}(\mathcal{E}_2) \leq S(\rho(t)) \leq \overline{S}(\mathcal{E}_2) + S(p),$$

where $\overline{S}(\mathcal{E}_2) = pS(\rho_1) + (1-p)S(\rho_2)$ is the average entropy
and $S(p) = -p\ln p - (1-p)\ln(1-p)$ is a binary entropy.

A ***mixing rate*** is defined as

$$\Lambda(\mathcal{E}_2, H) = \left.\frac{dS(\rho(t))}{dt}\right|_{t=0}.$$

Conjectured by Bravyj '07:

Theorem

Small Incremental Mixing*.*
(Van Acoleyen et. al. '13) For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), (1-p, \rho_2)\}$*, the maximum mixing rate is bounded above by a binary Shannon entropy.*

$$\begin{aligned}\Lambda(\mathcal{E}_2) &:= c \max\{|\Lambda(\mathcal{E}_2, H)| : -I \leq H \leq I\} \\ &\leq c\, S(p) = c\{-p \ln p - (1-p)\ln(1-p)\}.\end{aligned}$$

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A stronger bound for $1/100 < p < 99/100$.

Theorem

(E. H. Lieb, A. V.' 13) For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), (1-p, \rho_2)\}$, the maximum mixing rate is bounded above

$$\Lambda(\mathcal{E}_2) \le 4\sqrt{p(1-p)}.$$

A Mixing Rate problem can be generalized for an ensemble consisting of any number of states.

SIM implies SIE

The entangling rate is

$$\Gamma(\Psi, H) = -i\mathrm{Tr}\Big(H_{AB}[\rho_{aAB}, \ln(\rho_{aA} \otimes \frac{I_B}{|B|})]\Big)$$

and the mixing rate is

$$\Lambda(\mathcal{E}_2, H) = -i\mathrm{Tr}(H\,[p\rho_1, \ln\rho]).$$

Lemma

(Braviy '07) For any mixed state ρ_{AB} there exists a mixed state μ_{AB} such that

$$\rho_A \otimes \frac{I_B}{|B|} = |B|^{-2}\rho_{AB} + (1 - |B|^{-2})\mu_{AB}.$$

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Define the ensemble $\mathcal{E}_2 = \{(|B|^{-2}, \rho_{AB}), (1 - |B|^{-2}, \mu_{AB})\}$. Then the average density state is $\tau_{AB} = \rho_A \otimes \frac{I_B}{|B|}$. Assuming SIM, we get

$$\Lambda(\mathcal{E}_2, H) \leq cS(|B|^{-2}) \leq 4c|B|^{-2}\ln|B|,$$

here we used $-x\ln x - (1-x)\ln(1-x) \leq 2x|\ln x|$. Therefore $\Gamma(\Psi, H) \leq 4c\ln|B|$. So SIM with const c implies SIE with const $4c$.

Entanglement rates in open systems

	Alice		**Bob**
systems	A	$\leftarrow \mathcal{L}_{AB} \rightarrow$	B
ancillas	a		b

initial state

$$\rho(0) = |\Psi\rangle\langle\Psi|_{aABb}$$

Time evolution of a state ρ for open system is the solution to

$$\frac{d\rho(t)}{dt} = \mathcal{L}_{AB}(\rho(t))$$

with the generator given by Hamiltonian and a term of Lindblad type

$$\mathcal{L}_{AB}(\rho) = -i[H_{AB}, \rho] + \sum_a L_{AB}(a)\rho L_{AB}^*(a) - \frac{1}{2}\Big(L_{AB}^*(a)L_{AB}(A)\rho + \rho L_{AB}^*(a)L_{AB}(a)\Big).$$

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The entanglement measure $E(\cdot)$ should satisfy the following assumptions:

1. E vanishes on product states
2. E is invariant under local unitary operations
3. E can not increase under LOCC operations

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The entanglement measure $E(\cdot)$ should satisfy the following assumptions:

1. E vanishes on product states
2. E is invariant under local unitary operations
3. E can not increase under LOCC operations

If $E(\rho(t))$ is differentiable, the **entangling rate** is

$$\Gamma(\Psi, \mathcal{L}) = \left.\frac{d\,E(\rho(t))}{dt}\right|_{t=0}.$$

For entanglement measure E the **entangling rate** for time $\Delta t > 0$ is

$$\Gamma(\Psi, \mathcal{L}, \Delta t) = \frac{E(\rho(\Delta t)) - E(\rho(0))}{\Delta t}.$$

Relative entropy of entanglement in ancilla-free system

Suppose that $d_B \leq d_A$ and $d_a = d_b = 1$.
A **relative entropy of entanglement** of a state $\rho_{AB}(t)$ is given by

$$D(\rho(t)) := \min_{\sigma\, sep} D(\rho(t)||\sigma) = \min_{\sigma\, sep} \mathrm{Tr}\Big(\rho(t) \ln \rho(t) - \rho(t) \log \sigma\Big),$$

where $\sigma_{AB} = \sum_j \alpha_j \sigma_A(j) \otimes \sigma_B(j)$ with $\sum_j \alpha_j = 1$. For pure states the relative entropy of entanglement is an entropy of entanglement.

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where $\sigma_{AB} = \sum_j \alpha_j \sigma_A(j) \otimes \sigma_B(j)$ with $\sum_j \alpha_j = 1$. For pure states the relative entropy of entanglement is an entropy of entanglement.

Theorem

(V. '15) For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$ the entangling rate for the relative entropy of entanglement has the following upper bound

$$\Gamma_R(\Psi, \mathcal{L}, \Delta t) \leq 4\Big(\|H\| + 86 \sum_\alpha \|L_\alpha\|^2\Big) \log d + \epsilon,$$

where $d = \min(d_A, d_B)$.

Beginning of the Proof

For state $|\Psi\rangle_{AB}$ with Schmidt decomposition

$$|\Psi\rangle = \sum_{n=1}^{d} \sqrt{p_n}\, |\phi_n\rangle_A\, |\psi_n\rangle_B\,.$$

the relative entropy of entanglement is achieved by a state

$$\sigma_0 = \sum_{n=1}^{d} p_n\, |\phi_n\rangle\langle\phi_n| \otimes |\psi_n\rangle\langle\psi_n|\,.$$

Proposition

(V. '15) For states $\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB}$ and σ_0 defined above, there exists a mixed state μ_{AB} such that

$$\sigma_0 = \frac{1}{d}\rho_{AB} + (1 - \frac{1}{d})\mu_{AB}.$$

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$$\sigma_0 = \frac{1}{d}\rho_{AB} + (1 - \frac{1}{d})\mu_{AB}.$$

At time $t = 0$: $D(\rho) = D(\rho||\sigma_0) = E(\Psi)$ with σ_0 discussed before.

For any time t: $D(\rho(t)) \leq D(\rho(t)||\sigma_0)$.

Therefore for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$

$$\Gamma_R(\Psi, \mathcal{L}, \Delta t) \leq \frac{d}{dt} D(\rho(t)||\sigma_0)\Big|_{t=0} + \epsilon.$$

The derivative of relative entropy is calculated as follows, for $p = 1/d$,

$$
\begin{aligned}
\frac{d}{dt} D(\rho(t)||\sigma_0)\Big|_{t=0} &= \mathrm{Tr}(\dot\rho(0)\log\rho - \dot\rho(0)\log\sigma_0) \\
=& \frac{1}{p} i \mathrm{Tr}\Big(H[p\rho, \log(p\rho + (1-p)\mu)]\Big) \\
&- \frac{1}{2p}\sum_\alpha \mathrm{Tr}\Big(L_\alpha^*[L_\alpha(p\rho), \log(p\rho + (1-p)\mu]\Big) \\
&+ \frac{1}{2p}\sum_\alpha \mathrm{Tr}\Big(L_\alpha[(p\rho)L_\alpha^*, \log(p\rho + (1-p)\mu)]\Big) - \sum_\alpha \mathrm{Tr}(L_\alpha^*[L_\alpha\rho, \log\rho]).
\end{aligned}
$$

Each term can be made of the form

$$|\mathrm{Tr}(\tilde{L}^*[\tilde{L}X, \log Y])|,$$

where $\|\tilde{L}\| = 1$, $0 \leq X \leq Y \leq I$ and $\mathrm{Tr}\, Y = 1$, $\mathrm{Tr} X = p$.

Lemma

(V. '15) For $0 \leq X \leq Y \leq I$*, TrY* $= 1$*, TrX* $= p$ *and* $\|\tilde{L}\| = 1$*,*

$$|Tr(\tilde{L}^*[\tilde{L}X, \log Y])| \leq 172\, p \log(1/p).$$

□

Quantum Mutual Information - ancilla-assisted case

The quantum mutual information of a state ρ_{aABb} in a bipartite cut Alice$-$Bob is:

$$I(aA;Bb)_\rho = S(\rho_{aA}) + S(\rho_{Bb}) - S(\rho_{aABb}) = D(\rho_{aABb}||\rho_{aA} \otimes \rho_{Bb}).$$

Theorem

(V. '15) For a system starting in pure state $\rho_{aABb} = |\Psi\rangle\langle\Psi|_{aABb}$ and evolving with generator $\mathcal{L}$ the following holds

$$\frac{d}{dt} I(aA;Bb)_{\rho(t)}\Big|_{t=0} \leq 4\Big(2\|H\| + 129\sum_\alpha \|L_\alpha\|^2\Big)(\log d_A + \log d_B).$$

Open Questions

Question

Small incremental entangling in open system (V. '15).
Denote $d = \min\{d_A, d_B\}$. For which entanglement measures there exists a constant c and a non-negative and non-decreasing function $f(\cdot)$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$ the entangling rate is bounded above by

$$\Gamma(\Psi, \mathcal{L}, \Delta t) \leq c\|\mathcal{L}\| f(d) + \epsilon,$$

where c is independent of the dimensions of systems A and B, ancillas a, b, the generator $\mathcal{L}$ and the initial state $|\Psi\rangle_{aABb}$.

- Small Incremental Entangling Problem for
  - Renyi entropies
  - Entanglement of Formation
  - Negativity
  - ...
- Stability of Area Law for open systems
- SIE for multipartite systems

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Thank you!