

Entanglement Rates in Bipartite Systems

Anna Vershynina

Department of Mathematics, TU Munich, Germany

Mathematical Challenges in Quantum Mechanics

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Outline of the talk

- Entangling Rate in Closed Systems
- Mixing Rate Problem
- Entangling Rate in Open Systems
 - Relative Entropy of Entanglement in ancilla-free system
 - Quantum Mutual Information in ancilla-assisted system
- Future Research

Entangling Rate in Closed System

	Alice		Bob
systems	A	$\leftarrow H_{AB} \rightarrow$	B
ancillas	a		b
		initial state	
		$\rho(0) = \Psi\rangle\langle\Psi _{aABb}$	

Time evolution in Schrödinger picture

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] = -i[\mathbb{1}_a \otimes H_{AB} \otimes \mathbb{1}_b, \rho(t)]$$

has an explicit solution

$$\rho(t) = U^*(t) |\Psi\rangle\langle\Psi| U(t),$$

where

$$U(t) = e^{itH} = I_a \otimes e^{itH_{AB}} \otimes I_b$$

is a unitary evolution. The state $\rho(t)$ is always pure.

To measure the entanglement between Alice and Bob, we calculate the **entanglement entropy**

$$E(\rho(t)) := S(\rho_{aA}(t)) = -\text{Tr} \rho_{aA}(t) \ln \rho_{aA}(t),$$

here $\rho_{aA}(t) = \text{Tr}_{Bb} \rho_{aABb}(t) = \text{Tr}_{Bb} U^*(t) |\Psi\rangle \langle \Psi| U(t)$.

Remark

Small Total Entangling (Bennet et al) The total change of the entanglement $E(\rho(t))$ is at most $2 \ln d$, where $d = \min\{|A|, |B|\}$.

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The **entangling rate** is defined by

$$\Gamma(\Psi, H) = \left. \frac{dE(\rho(t))}{dt} \right|_{t=0}.$$

It can be expressed as

$$\Gamma(\Psi, H) = -i \text{Tr} \left(H_{AB} [\rho_{aAB}, \ln(\rho_{aA}) \otimes I_B] \right).$$

Conjectured by Bravyi '07:

Theorem

Small Incremental Entangling

There is a universal constant c such that for all dimensions of ancillas a, b and for all states $|\Psi\rangle$, the following holds

$$\Gamma(\Psi, H) \leq c\|H\| \ln d,$$

where $d = \min\{|A|, |B|\}$.

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History of results:

- 2003 Bennett et al.: $\Gamma(\Psi, H) \leq c\|H\|d^4$
- 2007 Bravyi: $\Gamma(\Psi, H) \leq 2\|H\|d^2$
- 2007 Bravyi, no ancillas: $\Gamma(\Psi, H) \leq c(d)\|H\| \log d$, with $c(d) \rightarrow 1$ with large d
- 2013 Lieb, Vershynina: $\Gamma(\Psi, H) \leq (4/\ln 2)\|H\|d$
- 2013 Van Acoleyen, Mariën, Verstraete: $\Gamma(\Psi, H) \leq 18\|H\| \log d$
- 2013 Audenaert: $\Gamma(\Psi, H) \leq 8\|H\| \log d$

Mixing Rate Problem

\mathcal{H} is a Hilbert space of dimension d . Let $\mathcal{E}_2 = \{(p, \rho_1), ((1 - p), \rho_2)\}$ be a probabilistic ensemble on \mathcal{H} with expected density operator

$$\rho = p\rho_1 + (1 - p)\rho_2.$$

For any Hamiltonian H the time-dependent state is

$$\rho(t) = p\rho_1 + (1 - p)e^{-itH}\rho_2e^{itH}.$$

The von Neumann entropy of this state is

$$S(\rho(t)) = -\text{Tr} \rho(t) \ln \rho(t).$$

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Small Total Mixing For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), ((1 - p), \rho_2)\}$, the entropy of a state $\rho(t)$ at any time t satisfies

$$\bar{S}(\mathcal{E}_2) \leq S(\rho(t)) \leq \bar{S}(\mathcal{E}_2) + S(p),$$

where $\bar{S}(\mathcal{E}_2) = pS(\rho_1) + (1 - p)S(\rho_2)$ is the average entropy and $S(p) = -p \ln p - (1 - p) \ln(1 - p)$ is a binary entropy.

A **mixing rate** is defined as

$$\Lambda(\mathcal{E}_2, H) = \left. \frac{dS(\rho(t))}{dt} \right|_{t=0}.$$

Conjectured by Bravyj '07:

Theorem

Small Incremental Mixing.

(Van Acoleyen et. al. '13) For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), (1 - p, \rho_2)\}$, the maximum mixing rate is bounded above by a binary Shannon entropy.

$$\begin{aligned} \Lambda(\mathcal{E}_2) &:= c \max\{|\Lambda(\mathcal{E}_2, H)| : -I \leq H \leq I\} \\ &\leq c S(p) = c\{-p \ln p - (1 - p) \ln(1 - p)\}. \end{aligned}$$

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A stronger bound for $1/100 < p < 99/100$.

Theorem

(E. H. Lieb, A. V. '13) For any ensemble $\mathcal{E}_2 = \{(p, \rho_1), (1 - p, \rho_2)\}$, the maximum mixing rate is bounded above

$$\Lambda(\mathcal{E}_2) \leq 4\sqrt{p(1 - p)}.$$

A Mixing Rate problem can be generalized for an ensemble consisting of any number of states.

SIM implies SIE

The entangling rate is

$$\Gamma(\Psi, H) = -i\text{Tr}\left(H_{AB}[\rho_{aAB}, \ln(\rho_{aA} \otimes \frac{I_B}{|B|})]\right)$$

and the mixing rate is

$$\Lambda(\mathcal{E}_2, H) = -i\text{Tr}(H[\rho\rho_1, \ln \rho]).$$

Lemma

(Bravyi '07) For any mixed state ρ_{AB} there exists a mixed state μ_{AB} such that

$$\rho_A \otimes \frac{I_B}{|B|} = |B|^{-2}\rho_{AB} + (1 - |B|^{-2})\mu_{AB}.$$

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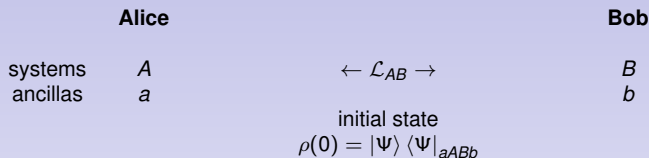
$$\rho_A \otimes \frac{I_B}{|B|} = |B|^{-2} \rho_{AB} + (1 - |B|^{-2}) \mu_{AB}.$$

Define the ensemble $\mathcal{E}_2 = \{(|B|^{-2}, \rho_{AB}), (1 - |B|^{-2}, \mu_{AB})\}$. Then the average density state is $\tau_{AB} = \rho_A \otimes \frac{I_B}{|B|}$. Assuming SIM, we get

$$\Lambda(\mathcal{E}_2, H) \leq cS(|B|^{-2}) \leq 4c|B|^{-2} \ln |B|,$$

here we used $-x \ln x - (1-x) \ln(1-x) \leq 2x |\ln x|$. Therefore $\Gamma(\Psi, H) \leq 4c \ln |B|$. So SIM with const c implies SIE with const $4c$.

Entanglement rates in open systems



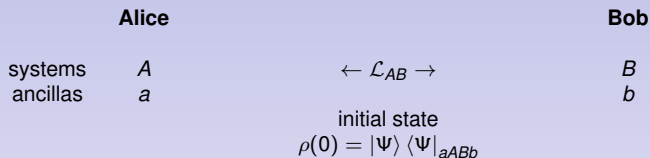
Time evolution of a state ρ for open system is the solution to

$$\frac{d\rho(t)}{dt} = \mathcal{L}_{AB}(\rho(t))$$

with the generator given by Hamiltonian and a term of Lindblad type

$$\mathcal{L}_{AB}(\rho) = -i[H_{AB}, \rho] + \sum_a L_{AB}(a)\rho L_{AB}^*(a) - \frac{1}{2}\left(L_{AB}^*(a)L_{AB}(A)\rho + \rho L_{AB}^*(a)L_{AB}(a)\right).$$

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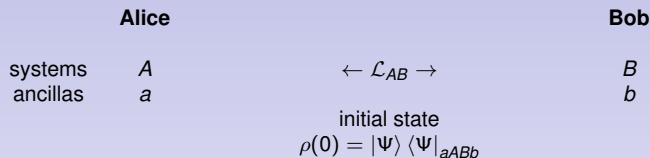
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The entanglement measure $E(\cdot)$ should satisfy the following assumptions:

- 1 E vanishes on product states
- 2 E is invariant under local unitary operations
- 3 E can not increase under LOCC operations

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The entanglement measure $E(\cdot)$ should satisfy the following assumptions:

- 1 E vanishes on product states
- 2 E is invariant under local unitary operations
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If $E(\rho(t))$ is differentiable, the **entangling rate** is

$$\Gamma(\Psi, \mathcal{L}) = \left. \frac{dE(\rho(t))}{dt} \right|_{t=0}.$$

For entanglement measure E the **entangling rate** for time $\Delta t > 0$ is

$$\Gamma(\Psi, \mathcal{L}, \Delta t) = \frac{E(\rho(\Delta t)) - E(\rho(0))}{\Delta t}.$$

Relative entropy of entanglement in ancilla-free system

Suppose that $d_B \leq d_A$ and $d_a = d_b = 1$.

A **relative entropy of entanglement** of a state $\rho_{AB}(t)$ is given by

$$D(\rho(t)) := \min_{\sigma_{sep}} D(\rho(t) || \sigma) = \min_{\sigma_{sep}} \text{Tr} \left(\rho(t) \ln \rho(t) - \rho(t) \log \sigma \right),$$

where $\sigma_{AB} = \sum_j \alpha_j \sigma_A(j) \otimes \sigma_B(j)$ with $\sum_j \alpha_j = 1$. For pure states the relative entropy of entanglement is an entropy of entanglement.

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Theorem

(V. '15) For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$ the entangling rate for the relative entropy of entanglement has the following upper bound

$$\Gamma_R(\Psi, \mathcal{L}, \Delta t) \leq 4 \left(\|H\| + 86 \sum_{\alpha} \|L_{\alpha}\|^2 \right) \log d + \epsilon,$$

where $d = \min(d_A, d_B)$.

Beginning of the Proof

For state $|\Psi\rangle_{AB}$ with Schmidt decomposition

$$|\Psi\rangle = \sum_{n=1}^d \sqrt{\rho_n} |\phi_n\rangle_A |\psi_n\rangle_B.$$

the relative entropy of entanglement is achieved by a state

$$\sigma_0 = \sum_{n=1}^d \rho_n |\phi_n\rangle \langle \phi_n| \otimes |\psi_n\rangle \langle \psi_n|.$$

Proposition

(V. '15) For states $\rho_{AB} = |\Psi\rangle \langle \Psi|_{AB}$ and σ_0 defined above, there exists a mixed state μ_{AB} such that

$$\sigma_0 = \frac{1}{d} \rho_{AB} + \left(1 - \frac{1}{d}\right) \mu_{AB}.$$

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$$\sigma_0 = \frac{1}{d} \rho_{AB} + \left(1 - \frac{1}{d}\right) \mu_{AB}.$$

At time $t = 0$: $D(\rho) = D(\rho||\sigma_0) = E(\Psi)$ with σ_0 discussed before.

For any time t : $D(\rho(t)) \leq D(\rho(t)||\sigma_0)$.

Therefore for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$

$$\Gamma_R(\Psi, \mathcal{L}, \Delta t) \leq \left. \frac{d}{dt} D(\rho(t)||\sigma_0) \right|_{t=0} + \epsilon.$$

The derivative of relative entropy is calculated as follows, for $p = 1/d$,

$$\begin{aligned} \left. \frac{d}{dt} D(\rho(t) || \sigma_0) \right|_{t=0} &= \text{Tr}(\dot{\rho}(0) \log \rho - \dot{\rho}(0) \log \sigma_0) \\ &= \frac{1}{p} i \text{Tr} \left(H[p\rho, \log(p\rho + (1-p)\mu)] \right) \\ &\quad - \frac{1}{2p} \sum_{\alpha} \text{Tr} \left(L_{\alpha}^* [L_{\alpha}(p\rho), \log(p\rho + (1-p)\mu)] \right) \\ &\quad + \frac{1}{2p} \sum_{\alpha} \text{Tr} \left(L_{\alpha} [(p\rho)L_{\alpha}^*, \log(p\rho + (1-p)\mu)] \right) - \sum_{\alpha} \text{Tr} (L_{\alpha}^* [L_{\alpha}\rho, \log \rho]). \end{aligned}$$

Each term can be made of the form

$$|\text{Tr}(\tilde{L}^* [\tilde{L} X, \log Y])|,$$

where $\|\tilde{L}\| = 1$, $0 \leq X \leq Y \leq I$ and $\text{Tr} Y = 1$, $\text{Tr} X = p$.

Lemma

(V. '15) For $0 \leq X \leq Y \leq I$, $\text{Tr} Y = 1$, $\text{Tr} X = p$ and $\|\tilde{L}\| = 1$,

$$|\text{Tr}(\tilde{L}^* [\tilde{L} X, \log Y])| \leq 172 p \log(1/p).$$

□

Quantum Mutual Information - ancilla-assisted case

The quantum mutual information of a state ρ_{aABb} in a bipartite cut Alice–Bob is:

$$I(aA; Bb)_\rho = S(\rho_{aA}) + S(\rho_{Bb}) - S(\rho_{aABb}) = D(\rho_{aABb} || \rho_{aA} \otimes \rho_{Bb}).$$

Theorem

(V. '15) For a system starting in pure state $\rho_{aABb} = |\Psi\rangle\langle\Psi|_{aABb}$ and evolving with generator \mathcal{L} the following holds

$$\left. \frac{d}{dt} I(aA; Bb)_{\rho(t)} \right|_{t=0} \leq 4 \left(2\|H\| + 129 \sum_{\alpha} \|L_{\alpha}\|^2 \right) (\log d_A + \log d_B).$$

Question

Small incremental entangling in open system (V. '15).

Denote $d = \min\{d_A, d_B\}$. For which entanglement measures there exists a constant c and a non-negative and non-decreasing function $f(\cdot)$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta t < \delta$ the entangling rate is bounded above by

$$\Gamma(\Psi, \mathcal{L}, \Delta t) \leq c \|\mathcal{L}\| f(d) + \epsilon,$$

where c is independent of the dimensions of systems A and B , ancillas a, b , the generator \mathcal{L} and the initial state $|\Psi\rangle_{aABb}$.

- Small Incremental Entangling Problem for
 - Renyi entropies
 - Entanglement of Formation
 - Negativity
 - ...
- Stability of Area Law for open systems
- SIE for multipartite systems

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Thank you!